

WEAK POSITIVITY THEOREM AND FROBENIUS STABLE CANONICAL RINGS OF GEOMETRIC GENERIC FIBERS

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ABSTRACT. In this paper, we prove the weak positivity theorem in positive characteristic when the canonical ring of the geometric generic fiber F is finitely generated and the Frobenius stable canonical ring of F is large enough. As its application, we show the subadditivity of Kodaira dimensions in some new cases.

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1. INTRODUCTION

Let $f : X \rightarrow Y$ be a separable surjective morphism between smooth projective varieties over an algebraically closed field satisfying $f_* \mathcal{O}_X \cong \mathcal{O}_Y$. The positivity of the direct image sheaf $f_* \omega_{X/Y}^m$ of the relative pluricanonical bundle is an important property. In characteristic zero, there are numerous known results. Fujita proved that $f_* \omega_{X/Y}$ is a nef vector bundle when $\dim Y = 1$ [Fuj78]. Kawamata generalized this to the case when $m \geq 2$ [Kaw82] and to the case when $\dim Y \geq 2$ [Kaw81] (see also [Fuj14]). Viehweg showed that $f_* \omega_{X/Y}^m$ is weakly positive for each $m \geq 1$ [Vie83] (see also [Kol87], [Cam04], and [Fuj13]). Here weak positivity is a property of coherent sheaves, which can be viewed as a generalization of nefness of vector bundles, and is equivalent to nefness when we restrict ourselves to vector bundles on smooth projective curves. There are several significant consequences of these results. One of them is Iitaka's conjecture in some special cases. Iitaka's conjecture states that the subadditivity of Kodaira dimensions

$$\kappa(X) \geq \kappa(Y) + \kappa(X_{\bar{\eta}})$$

holds, where $X_{\bar{\eta}}$ is the geometric generic fiber of f (note that this conjecture follows from a conjecture in the minimal model program [Kaw85]). Other consequences include some moduli problems in [Kol90] and [Fuj12] (see also [Vie95]), where results

of [Fuj78], [Kaw81], and [Kaw82] are generalized to the case when X is reducible (see also [Kaw11], [FF14], and [FFS14]).

On the other hand, in positive characteristic, it is known that there are counter-examples to the above results. For example, Moret-Bailly constructed a semi-stable fibration $g : S \rightarrow \mathbb{P}^1$ from a surface S to \mathbb{P}^1 such that $g_*\omega_{S/\mathbb{P}^1}$ is not nef [MB81]. For other examples, see [Ray78, Xie10] (or Remark 5.2 in this paper). Hence it is natural to ask under what additional conditions analogous results hold in positive characteristic. Kollar showed that $f_*\omega_{X/Y}^m$ is a nef vector bundle for each $m \geq 2$ when X is a surface, Y is a curve, and the general fiber of f has only nodes as singularities [Kol90, 4.3. Theorem]. Patakfalvi proved that $f_*\omega_{X/Y}^m$ is a nef vector bundle for each $m \gg 0$ when Y is a curve, $X_{\bar{\eta}}$ has only normal F -pure singularities, and $\omega_{X/Y}$ is f -ample [Pat14, Theorem 1.1].

In this paper, we consider the weak positivity of $f_*\omega_{X/Y}^m$ in positive characteristic under a condition on the canonical ring and the Frobenius stable canonical ring of the geometric generic fiber. Recall that for a Gorenstein variety V , the canonical ring of V is the section ring of the dualizing sheaf of V , and the Frobenius stable canonical ring of V is its homogeneous ideal whose degree m subgroup is $S^0(V, mK_V)$. $S^0(V, mK_V)$ is the subspace of $H^0(V, mK_V)$ defined by using the trace map of the Frobenius morphism (see Definition 3.2 or [Sch14, §4]). These notions are naturally extended to pairs (V, Δ) consisted of Gorenstein varieties V and effective $\mathbb{Z}_{(p)}$ -Cartier divisors Δ on V . We use this setting throughout this paper.

From now on we work over an algebraically closed field of characteristic $p > 0$. The following theorem is a main result of this paper.

Theorem 1.1 (Theorem 5.1). *Let $f : X \rightarrow Y$ be a separable surjective morphism between smooth projective varieties, let Δ be an effective \mathbb{Q} -divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p , and let $\bar{\eta}$ be the geometric generic point of Y . Assume that*

- (i) *the $k(\bar{\eta})$ -algebra $\bigoplus_{m \geq 0} H^0(X_{\bar{\eta}}, m(aK_{X_{\bar{\eta}}} + (a\Delta)_{\bar{\eta}}))$ is finitely generated, and*
- (ii) *there exists an integer $m_0 > 0$ such that for each $m \geq m_0$,*

$$S^0(X_{\bar{\eta}}, \Delta_{\bar{\eta}}, m(aK_{X_{\bar{\eta}}} + (a\Delta)_{\bar{\eta}})) = H^0(X_{\bar{\eta}}, m(aK_{X_{\bar{\eta}}} + (a\Delta)_{\bar{\eta}})).$$

Then $f_\mathcal{O}_X(am(K_{X/Y} + \Delta))$ is weakly positive for each $m \geq m_0$.*

Condition (ii) holds, for example, in the case where $X_{\bar{\eta}}$ is a curve of arithmetic genus at least two which has only nodes as singularities, $\Delta = 0$, and $m_0 = 2$ (Corollary 3.14), or in the case where the pair $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ has only F -pure singularities, $K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}$ is ample, and $m_0 \gg 0$ (Example 3.11). Thus Theorem 1.1 is a generalization of [Kol90, 4.3. Theorem] and [Pat14, Theorem 1.1].

Theorem 1.1 should be compared with another result of Patakfalvi [Pat13, Theorem 6.4], which states that if $S^0(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}) = H^0(X_{\bar{\eta}}, K_{X_{\bar{\eta}}})$ then $f_*\omega_{X/Y}$ is weakly positive (see also [Jan08]). These two results imply that $S^0(X_{\bar{\eta}}, mK_{X_{\bar{\eta}}})$ is closely related to the positivity of $f_*\omega_{X/Y}^m$ for each $m \geq 1$. In order to prove Theorem 1.1, we generalize the method of the proof of [Pat13, Theorem 6.4] using a numerical invariant introduced in Section 4.

When the relative dimension of f is one, we obtain the following theorem as a corollary of Theorem 1.1.

Theorem 1.2 (Corollary 5.4). *Let $f : X \rightarrow Y$ be a separable surjective morphism of relative dimension one between smooth projective varieties satisfying $f_* \mathcal{O}_X \cong \mathcal{O}_Y$, let Δ be an effective \mathbb{Q} -divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p , and let $\bar{\eta}$ be the geometric generic point of Y . If $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is F -pure and $K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}$ is ample, then $f_* \mathcal{O}_X(am(K_{X/Y} + \Delta))$ is weakly positive for each $m \geq 2$. In particular, if $X_{\bar{\eta}}$ is smooth curve of genus at least two, then $f_* \omega_{X/Y}^m$ is weakly positive for each $m \geq 2$.*

Moreover, in the case where $f : X \rightarrow Y$ is a semi-stable fibration from a surface to a curve, we discuss the ampleness of $f_* \omega_{X/Y}^m$ for each $m \geq 2$ and the nefness of $f_* \omega_{X/Y}$ (Theorems 6.8 and 6.13).

When the relative dimension of f is two, we also obtain the following theorem as a corollary of Theorem 1.1.

Theorem 1.3 (Corollary 5.5). *Let $f : X \rightarrow Y$ be a separable surjective morphism of relative dimension two between smooth projective varieties satisfying $f_* \mathcal{O}_X \cong \mathcal{O}_Y$. If the geometric generic fiber is a smooth surface of general type and $p \geq 7$, then $f_* \omega_{X/Y}^m$ is weakly positive for each $m \gg 0$.*

Similarly to the case of characteristic zero, we can use Theorem 1.1 to study Iitaka's conjecture. Before stating the next theorem, we recall the definition of Iitaka-Kodaira dimension. Let D be a Cartier divisor on a projective variety V and let $m > 0$ be an integer divisible enough. The Iitaka-Kodaira dimension $\kappa(V, D)$ of D is the dimension of the image of V under the rational map determined by a linear system $|mD|$ if it is not empty, otherwise $\kappa(V, D) = -\infty$. This definition is naturally generalized to the case where D is a $\mathbb{Z}_{(p)}$ -Cartier (or \mathbb{Q} -Cartier) divisor. When V is smooth, the Kodaira dimension $\kappa(V)$ of V is defined as $\kappa(V, K_V)$.

Theorem 1.4 (Theorems 7.2 and 7.6). *Let $f : X \rightarrow Y$ be a separable surjective morphism between smooth projective varieties satisfying $f_* \mathcal{O}_X \cong \mathcal{O}_Y$, let Δ be an effective \mathbb{Q} -divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p , and let $\bar{\eta}$ be the geometric generic point of Y . Assume that*

- (i) *the $k(\bar{\eta})$ -algebra $\bigoplus_{m \geq 0} H^0(X_{\bar{\eta}}, m(aK_{X_{\bar{\eta}}} + (a\Delta)_{\bar{\eta}}))$ is finitely generated,*
- (ii) *there exists an integer $m_0 \geq 0$ such that*

$$S^0(X_{\bar{\eta}}, \Delta_{\bar{\eta}}, m(aK_{X_{\bar{\eta}}} + (a\Delta)_{\bar{\eta}})) = H^0(X_{\bar{\eta}}, m(aK_{X_{\bar{\eta}}} + (a\Delta)_{\bar{\eta}}))$$

for each $m \geq m_0$, and

- (iii) *either that Y is of general type or Y is an elliptic curve.*

Then

$$\kappa(X, K_X + \Delta) \geq \kappa(Y) + \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}).$$

As a special case of Theorem 1.4, we obtain the following result.

Theorem 1.5 (Corollary 7.8). *Let $f : X \rightarrow Y$ be a separable surjective morphism from a smooth projective variety X of dimension three to a smooth projective curve Y satisfying $f_* \mathcal{O}_X \cong \mathcal{O}_Y$. If the geometric generic fiber $X_{\bar{\eta}}$ is a smooth projective surface of general type and $p \geq 7$, then*

$$\kappa(X) \geq \kappa(Y) + \kappa(X_{\bar{\eta}}).$$

1.1. Notation. In this paper, we fix an algebraically closed field k of characteristic $p > 0$. A k -scheme is a separated scheme of finite type over k . A *variety* means an integral k -scheme and a *curve* (resp. *surface*) means a variety of dimension one (resp. two). A projective surjective morphism $f : X \rightarrow Y$ between varieties is called a *fibration* or a *algebraic fiber space* if it is separable and it satisfies $f_* \mathcal{O}_X \cong \mathcal{O}_Y$.

We fix the following notation:

- Let $\text{CDiv}(S)$ be the group of the Cartier divisors on a scheme S . An $\mathbb{Z}_{(p)}$ -Cartier (resp. \mathbb{Q} -Cartier) divisor on S is an element of $\text{CDiv}(S) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ (resp. $\text{CDiv}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$), where $\mathbb{Z}_{(p)}$ is the localization of \mathbb{Z} at the prime ideal $(p) = p\mathbb{Z}$;
- For a rational number δ , we denote its integral (resp. fractional) part by $\lfloor \delta \rfloor$ (resp. $\{\delta\}$). For a \mathbb{Q} -divisor $\Delta = \sum_i \delta_i \Delta_i$ on a normal variety, we define $\lfloor \Delta \rfloor := \sum_i \lfloor \delta_i \rfloor \Delta_i$ (resp. $\{\Delta\} := \sum_i \{\delta_i\} \Delta_i$);
- Let $\varphi : S \rightarrow T$ be a morphism of schemes and let T' be a T -scheme. Then we denote the second projection $S_{T'} := S \times_T T' \rightarrow T'$ by $\varphi_{T'}$. For a Cartier (or $\mathbb{Z}_{(p)}$ -Cartier, \mathbb{Q} -Cartier) divisor D on S , the pullback of D to $S_{T'}$ is denoted by $D_{T'}$ if it is well-defined. Similarly, for an \mathcal{O}_S -module homomorphism $\alpha : \mathcal{F} \rightarrow \mathcal{G}$, the pullback of α to $S_{T'}$ is denoted by $\alpha_{T'} : \mathcal{F}_{T'} \rightarrow \mathcal{G}_{T'}$;
- For a scheme X of positive characteristic, $F_X : X \rightarrow X$ is the absolute Frobenius morphism. We often denote the source of F_X^e by X^e . Let $f : X \rightarrow Y$ be a morphism between schemes of positive characteristic. We denote the same morphism by $f^{(e)} : X^e \rightarrow Y^e$ when we regard X (resp. Y) as X^e (resp. Y^e). We define the e -th relative Frobenius morphism of f to be the morphism $F_{X/Y}^{(e)} := (F_X^e, f^{(e)}) : X^e \rightarrow X \times_Y Y^e =: X_{Y^e}$.

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2. PRELIMINARIES

2.1. AC divisors. Let X be a k -scheme of pure dimension satisfying S_2 and G_1 . An *AC divisor* (or *almost Cartier divisor*) on X is a coherent \mathcal{O}_X -submodule of the sheaf of total quotient ring $K(X)$ which is invertible in codimension one (see [Kol+92], [Har94], or [MS12]). For any AC divisor D we denote the coherent sheaf defining D by $\mathcal{O}_X(D)$. The set of AC divisors $\text{WSh}(X)$ has the structure of additive group [Har94, Corollary 2.6]. A $\mathbb{Z}_{(p)}$ -*AC divisor* is an element of $\text{WSh}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. An AC divisor D is said to be *effective* if $\mathcal{O}_X \subseteq \mathcal{O}_X(D)$, and a $\mathbb{Z}_{(p)}$ -AC divisor Δ is said to be *effective* if $\Delta = D \otimes r$ for some effective AC divisor D and some

$0 \leq r \in \mathbb{Z}_{(p)}$. Now we have the following diagram:

$$\begin{array}{ccc} \mathrm{WSh}(X) & \xrightarrow{(\cup)^{\otimes 1}} & \mathrm{WSh}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ \mathrm{CDiv}(X) & \xrightarrow{(\cup)^{\otimes 1}} & \mathrm{CDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \end{array}$$

Note that the horizontal homomorphisms are not necessarily injective [Kol+92, Page 172]. **Throughout this paper, given an effective $\mathbb{Z}_{(p)}$ -AC (resp. $\mathbb{Z}_{(p)}$ -Cartier) divisor Δ , we fix an effective AC (resp. Cartier) divisor E and an integer $a > 0$ not divisible by p such that $E \otimes 1 = a\Delta$.** The choice of E and a is often represented by $\Delta = E/a$. For every integer m , we regard the $\mathbb{Z}_{(p)}$ -AC divisor $am\Delta$ as the AC divisor mE . For instance, the symbol $\mathcal{O}_X(am(D + \Delta))$ denotes the sheaf $\mathcal{O}_X(amD + mE)$, for every AC divisor D .

We note that if X is a normal variety, then AC divisors are Weil divisors, and the horizontal homomorphisms in the above diagram is injective. In this case we can choose E and a canonically for an effective $\mathbb{Z}_{(p)}$ -divisors Δ : a is the smallest positive integer such that $a\Delta$ is integral and $E := a\Delta$.

We define notions similar to the above by using \mathbb{Q} instead of $\mathbb{Z}_{(p)}$.

2.2. Trace of Frobenius morphisms. In this subsection we introduce notations related to trace maps of Frobenius morphisms.

Let $\pi : X \rightarrow Y$ be a finite surjective morphism between Gorenstein k -schemes of pure dimension, and let ω_X and ω_Y be dualizing sheaves of X and Y respectively. We denote by $\mathrm{Tr}_{\pi} : \pi_* \omega_X \rightarrow \omega_Y$ the morphism obtained by applying the functor $\mathcal{H}om_{\mathcal{O}_Y}(_, \omega_Y)$ to the natural morphism $\pi^{\#} : \mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_X$. This is called the trace map of π . Then we define

$$\begin{aligned} \phi_X^{(1)} &:= \mathrm{Tr}_{F_X} \otimes \mathcal{O}_X(-K_X) : F_{X*} \mathcal{O}_X((1-p)K_X) \rightarrow \mathcal{O}_X, \quad \text{and} \\ \phi_X^{(e+1)} &:= \phi_X^{(e)} \circ F_{X*}^e (\phi_X^{(1)} \otimes \mathcal{O}_X((1-p^e)K_X)) : F_X^{e+1} \mathcal{O}_X((1-p^{e+1})K_X) \rightarrow \mathcal{O}_X \end{aligned}$$

for each $e > 0$, where K_X is a Cartier divisor satisfying $\mathcal{O}_X(K_X) \cong \omega_X$.

Let X be a Gorenstein k -scheme of pure dimension. Let $\Delta = E/a$ be an effective $\mathbb{Z}_{(p)}$ -Cartier divisor on X and $d > 0$ be the smallest integer satisfying $a|(p^d - 1)$. For each $e > 0$ we define

$$\begin{aligned} \mathcal{L}_{(X,\Delta)}^{(de)} &:= \mathcal{O}_X((1-p^{de})(K_X + \Delta)) \subseteq \mathcal{O}_X((1-p^{de})K_X), \\ \phi_{(X,\Delta)}^{(d)} &: F_X^d \mathcal{L}_{(X,\Delta)}^{(d)} \rightarrow F_X^d \mathcal{O}_X((1-p^d)K_X) \xrightarrow{\phi_X^{(d)}} \mathcal{O}_X, \quad \text{and} \\ \phi_{(X,\Delta)}^{(d(e+1))} &:= \phi_{(X,\Delta)}^{(de)} \circ F_X^{de} (\phi_{(X,\Delta)}^{(d)} \otimes \mathcal{L}_{(X,\Delta)}^{(de)}) : F_X^{d(e+1)} \mathcal{L}_{(X,\Delta)}^{(d(e+1))} \rightarrow \mathcal{O}_X. \end{aligned}$$

Let X be a k -scheme of pure dimension satisfying S_2 and G_1 . Let $\Delta = E/a$ be a $\mathbb{Z}_{(p)}$ -AC divisor on X and $d > 0$ be the smallest integer satisfying $a|(p^d - 1)$. Let $\iota : U \hookrightarrow X$ be a Gorenstein open subset of X such that $\mathrm{codim} X \setminus U \geq 2$ and that $E|_U$ is Cartier. Set $\Delta|_U = E|_U/a$. Then for each $e > 0$ we define

$$\mathcal{L}_{(X,\Delta)}^{(de)} := \iota_* \mathcal{L}_{(U,\Delta|_U)}^{(de)} \quad \text{and} \quad \phi_{(X,\Delta)}^{(de)} := \iota_* (\phi_{(U,\Delta|_U)}^{(de)}) : F_X^{de} \mathcal{L}_{(X,\Delta)}^{(de)} \rightarrow \mathcal{O}_X$$

Note that $\phi_{(X,\Delta)}^{(de)}$ is a morphism between reflexive sheaves on X (cf. [Har94, Proposition 1.11]).

Definition 2.1 ([Smi00, 2.1. Definition], [SS10, Definition 3.1] or [MS12, Definition 2.6]). With the notation as above, the pair (X, Δ) is said to be *sharply F -pure* (resp. *globally F -split*) if $\phi_{(X,\Delta)}^{(e)}$ is surjective (resp. split as \mathcal{O}_X -module homomorphism) for some $e > 0$ satisfying $a|(p^e - 1)$. We simply say that X is F -pure (resp. globally F -split) if $(X, 0)$ is F -pure (resp. globally F -split).

Remark 2.2. (1) (X, Δ) is F -pure if and only if $\phi_{(X,\Delta)}^{(e)}$ is surjective for any $e > 0$ satisfying $a|(p^e - 1)$. Indeed, if $\phi_{(X,\Delta)}^{(e)}$ is surjective for an $e > 0$ with $a|(p^e - 1)$, then $(\phi_{(X,\Delta)}^{(e)} \otimes \mathcal{L}_{(X,\Delta)}^{(eg)})^{**}$ is also surjective for any $g > 0$, and so is $\phi_{(X,\Delta)}^{(e(g+1))}$. Here $(_)^{**}$ is the functor of the double dual. Let $e' > 0$ be an integer with $a|(p^{e'} - 1)$. Then $\phi_{(X,\Delta)}^{(e')}$ is surjective, since $\phi_{(X,\Delta)}^{(ee')}$ is surjective and factors through $\phi_{(X,\Delta)}^{(e')}$.
(2) The F -purity of (X, Δ) is independent of the choice of E and a satisfying $\Delta = E/a$. Let E' be an effective AC divisor such that $E' = a'\Delta$ for an integer $a' > 0$ not divisible by p . Let $g > 0$ be an integer such that $q := p^{eg} - 1$ satisfies $a|q$, $a'|q$ and $qa^{-1}E = qa'^{-1}E'$. Then we see that

$$\phi_{(X,E/a)}^{(eg)} \cong \phi_{(X,qa^{-1}E/q)}^{(eg)} \cong \phi_{(X,qa'^{-1}E'/q)}^{(eg)} \cong \phi_{(X,E'/a')}^{(eg)}.$$

Thus $\phi_{(X,E'/a')}^{(e')}$ is surjective.

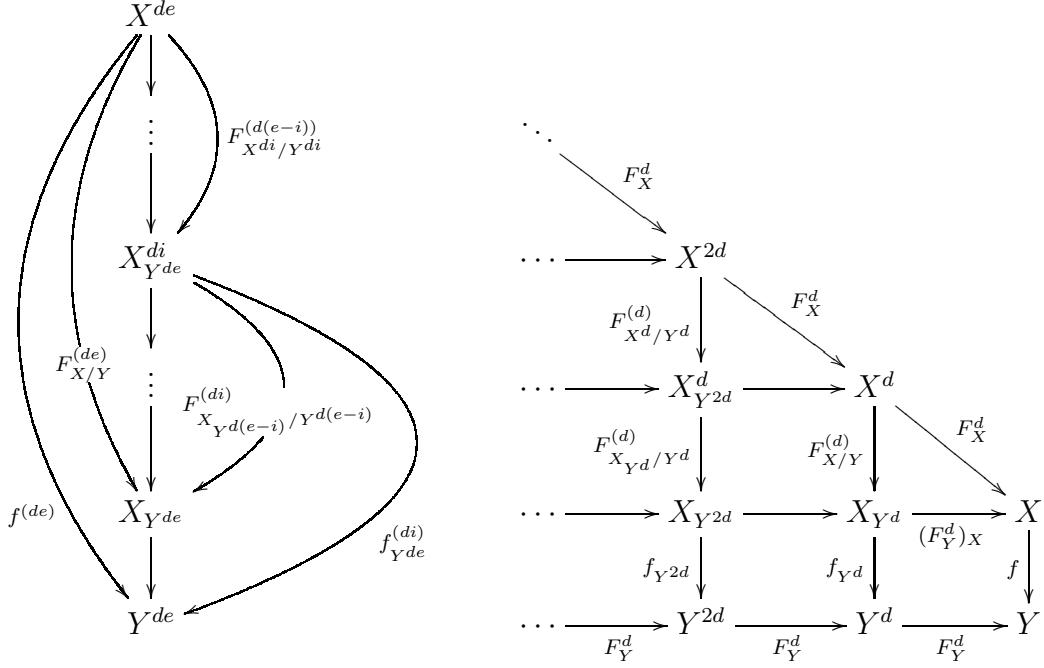
(3) A statement similar to (1) and (2) holds for the global F -splitting of (X, Δ) .

2.3. Trace of relative Frobenius morphisms. In this subsection, we introduce notations related to trace maps of relative Frobenius morphisms. See [PSZ13] for more details on trace maps of relative Frobenius morphisms.

Let $f : X \rightarrow Y$ be a morphism between Gorenstein k -schemes of pure dimension. We assume that either F_Y is flat (i.e., Y is regular) or f is flat. Then F_Y or f is a Gorenstein morphism, so X_{Y^1} is a Gorenstein k -scheme [Har66, III, §9]. We define the relative dualizing sheaf $\omega_{X/Y}$ of f to be $\omega_X \otimes f^* \omega_Y^{-1}$. Then we have

$$\begin{aligned} \omega_{X_{Y^1}/Y^1} &:= \omega_{X_{Y^1}} \otimes f_{Y^1}^* \omega_{Y^1}^{-1} \\ &\cong \omega_{X_{Y^1}} \otimes f_{Y^1}^* \omega_{Y^1}^{p-1} \otimes f_{Y^1}^* F_Y^* \omega_Y^{-1} \\ &\cong (F_Y)_X^! \mathcal{O}_X \otimes (F_Y)_X^* \omega_X \otimes (f_{Y^1}^* F_Y^! \mathcal{O}_Y)^{-1} \otimes (F_Y)_X^* f^* \omega_Y^{-1} \\ &\cong (F_Y)_X^* \omega_{X/Y} = (\omega_{X/Y})_{Y^1} \end{aligned}$$

by the assumption. Moreover, for positive integers d, e , we consider the following commutative diagram:



Let $K_{X/Y}$ be a Cartier divisor on X satisfying $\mathcal{O}_X(K_{X/Y}) \cong \omega_{X/Y}$. This is denoted by K_{X^e/Y^e} when we regard f as $f^{(e)}$. Set $K_{X_{Y^g}/Y^g} := (K_{X^e/Y^e})_{Y^g}$ for each $g \geq e$. Then for each $e > 0$ we define

$$\begin{aligned} \phi_{X/Y}^{(1)} &:= \text{Tr}_{F_{X/Y}^{(1)}} \otimes \mathcal{O}_{X_{Y^1}}(-K_{X_{Y^1}}) : F_{X/Y}^{(1)} \mathcal{O}_{X^1}((1-p)K_{X^1/Y^1}) \rightarrow \mathcal{O}_{X_{Y^1}}, \quad \text{and} \\ \phi_{X/Y}^{(e+1)} &:= \left(\phi_{X/Y}^{(e)} \right)_{Y^{e+1}} \circ F_{X_{Y^1}/Y^e}^{(e)} \left(\phi_{X^e/Y^e}^{(1)} \otimes \mathcal{O}_{X_{Y^{e+1}}}((1-p^e)K_{X_{Y^{e+1}}/Y^{e+1}}) \right) \\ &\quad : F_{X/Y}^{(e+1)} \mathcal{O}_X((1-p^{e+1})K_{X^{e+1}/Y^{e+1}}) \rightarrow \mathcal{O}_{X_{Y^{e+1}}}. \end{aligned}$$

Let $\Delta = E/a$ be an effective $\mathbb{Z}_{(p)}$ -AC divisor on X and d be the smallest positive integer satisfying $a|(p^d - 1)$. For each $e > 0$ we define

$$\begin{aligned} \mathcal{L}_{(X,\Delta)/Y}^{(de)} &:= \mathcal{O}_{X^{de}}((1-p^{de})(K_{X^{de}/Y^{de}} + \Delta)) \subseteq \mathcal{O}_{X^{de}}((1-p^{de})K_{X^{de}/Y^{de}}), \\ \phi_{(X,\Delta)/Y}^{(d)} &: F_{X/Y}^{(d)} \mathcal{L}_{(X,\Delta)/Y}^{(d)} \rightarrow F_{X/Y}^{(d)} \mathcal{O}_{X^d}((1-p^d)K_{X^d/Y^d}) \xrightarrow{\phi_{X/Y}^{(d)}} \mathcal{O}_{X_{Y^d}}, \quad \text{and} \\ \phi_{(X,\Delta)/Y}^{(d(e+1))} &:= \left(\phi_{(X,\Delta)/Y}^{(d)} \right)_{Y^{d(e+1)}} \circ F_{X_{Y^d}/Y^d}^{(de)} \left(\phi_{(X^{de},\Delta)/Y^{de}}^{(d)} \otimes \left(\mathcal{L}_{(X,\Delta)/Y}^{(de)} \right)_{Y^{d(e+1)}} \right) \\ &\quad : F_{X/Y}^{(d(e+1))} \mathcal{L}_{(X,\Delta)/Y}^{(d(e+1))} \rightarrow \mathcal{O}_{X_{Y^{d(e+1)}}}. \end{aligned}$$

Let $f : X \rightarrow Y$ be a morphism between k -schemes of pure dimension. Assume that X satisfies S_2 and G_1 , Y is Gorenstein, and f or F_Y is flat. Let $\Delta = E/a$ be an effective $\mathbb{Z}_{(p)}$ -AC divisor on X and d be the smallest positive integer satisfying $a|(p^d - 1)$. Let $\iota : U \hookrightarrow X$ be a Gorenstein open subset of X such that $\text{codim}X \setminus U \geq 2$

and that $E|_U$ is Cartier. Set $\Delta|_U = E|_U/a$. Then for each $e > 0$ we define

$$\begin{aligned}\mathcal{L}_{(X,\Delta)/Y}^{(de)} &:= \iota_{Y^{de}*}\mathcal{L}_{(U,\Delta|_U)/Y}^{(de)}, \quad \text{and} \\ \phi_{(X,\Delta)/Y}^{(de)} &:= \iota_{Y^{de}*}(\phi_{(U,\Delta|_U)}^{(de)}/Y) : F_{X/Y}^{(de)}\mathcal{L}_{(X,\Delta)/Y}^{(de)} \rightarrow \mathcal{O}_{X_{Y^{de}}}.\end{aligned}$$

3. FROBENIUS STABLE CANONICAL RING

In this section, we introduce and study Frobenius stable canonical rings. After definitions and basic properties, we study Frobenius stable canonical rings of varieties with ample canonical bundles. Especially, we consider the case of Gorenstein projective curves (Corollary 3.14). We also discuss the case of varieties with semi-ample canonical bundles in any dimension (Corollary 3.18). To this end, we prove Theorem 3.17 which is a kind of canonical bundle formula. As another application of the theorem, we study Frobenius stable canonical rings of surfaces of general type (Corollary 3.23).

Notation 3.1. Let X be a k -scheme of pure dimension satisfying S_2 and G_1 , and let $\Delta = E/a$ be an effective $\mathbb{Z}_{(p)}$ -AC divisor. Set $d > 0$ be the smallest integer satisfying $a|(p^d - 1)$.

Definition 3.2 ([Sch14, §3]). In the situation of Notation 3.1, let \mathcal{M} be a reflexive sheaf on X of rank one such that invertible in codimension one. Then we define $S^0(X, \Delta, \mathcal{M})$ as

$$\bigcap_{e>0} \text{im} \left(H^0(X, ((F_X^{de})_*\mathcal{L}_{(X,\Delta)}^{(de)}) \otimes \mathcal{M})^{**} \xrightarrow{H^0(X, (\phi_{(X,\Delta)}^{(de)} \otimes \mathcal{M})^{**})} H^0(X, \mathcal{M}) \right),$$

where $\phi_{(X,\Delta)}^{(de)}$ is the morphism defined in Subsection 2.2, and $(_)^{**} := \mathcal{H}\text{om}(\mathcal{H}\text{om}(_, \mathcal{O}_X), \mathcal{O}_X)$ is the functor of the double dual. For any AC divisor D on X , we denote $S^0(X, \Delta, \mathcal{O}_X(D))$ by $S^0(X, \Delta, D)$. Write $S^0(X, D) := S^0(X, 0, D)$.

Remark 3.3. The above definition does not depend on the choice of E and a satisfying $\Delta = E/a$. Indeed, if E' and a' satisfy $\Delta = E'/a'$, then by an argument similar to Remark 2.2 (2) we have $\phi_{(X,E/a)}^{(eg)} \cong \phi_{(X,E'/a')}^{(eg)}$ for every $g > 0$ divisible enough.

Example 3.4. In the situation of Notation 3.1, it is easily seen that the following are equivalent:

- (1) (X, Δ) is globally F -split.
- (2) $S^0(X, \Delta, \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$.
- (3) $S^0(X, \Delta, D) = H^0(X, D)$ for every AC divisor D on X .

Definition 3.5 ([HP13, Section 4.1] or [PST14, Exercise 4.13]). In the situation of Notation 3.1, let \mathcal{M} be a reflexive sheaf on X of rank one such that invertible in codimension one. Then we define

$$R_S(X, \Delta, \mathcal{M}) := \bigoplus_{n \geq 0} S^0(X, \Delta, (\mathcal{M}^{\otimes n})^{**}) \subseteq R(X, \mathcal{M}) := \bigoplus_{n \geq 0} H^0(X, (\mathcal{M}^{\otimes n})^{**}).$$

For any AC divisor D , we denote $R(X, \mathcal{O}_X(D))$ and $R_S(X, \Delta, \mathcal{O}_X(D))$ respectively by $R(X, D)$ and $R_S(X, \Delta, D)$. $R_S(X, \Delta, a(K_X + \Delta))$ is called the *Frobenius stable*

canonical ring, where K_X is an AC divisor such that $\mathcal{O}_X(K_X)$ is isomorphic to the dualizing sheaf ω_X of X .

When D is a \mathbb{Q} -Weil divisor on a normal variety X , we define

$$R_S(X, \Delta, D) := \bigoplus_{n \geq 0} S^0(X, \Delta, \lfloor nD \rfloor) \subseteq R(X, D) := \bigoplus_{n \geq 0} H^0(X, \lfloor nD \rfloor).$$

Lemma 3.6 ([HP13, Lemma 4.1.1]). $R_S(X, \Delta, D)$ is an ideal of $R(X, D)$.

Proof. This follows from an argument similar to the proof of [HP13, Lemma 4.1.1]. \square

Notation 3.7. We denote by $R/R_S(X, \Delta, D)$ the quotient ring of $R(X, D)$ modulo $R_S(X, \Delta, D)$.

We recall that the assumption (ii) of the main theorem (Theorem 1.1): there exists an $m_0 > 0$ such that $S^0(X_{\bar{\eta}}, \Delta_{\bar{\eta}}, am(K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}})) = H^0(X_{\bar{\eta}}, am(K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}))$ for every $m \geq m_0$. This is equivalent to the condition that there exists an integer $m_0 > 0$ such that the degree m part of $R/R_S(X_{\bar{\eta}}, \Delta_{\bar{\eta}}, a(K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}))$ is zero for every $m \geq m_0$. Note that the existence of such m_0 is equivalent to the finiteness of the dimension of k -vector space $R/R_S(X_{\bar{\eta}}, \Delta_{\bar{\eta}}, a(K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}))$.

Definition 3.8. In the situation of Notation 3.1, assume that each connected component of X is integral. An AC divisor D is said to be *finitely generated* if $R(X, D)$ is a finitely generated k -algebra. A $\mathbb{Z}_{(p)}$ -AC (resp. \mathbb{Q} -AC) divisor Γ is said to be *finitely generated* if there exists a finitely generated AC divisor D such that $\Gamma = D \otimes \lambda$ for some $0 < \lambda \in \mathbb{Z}_{(p)}$ (resp. \mathbb{Q}).

Lemma 3.9. Let $R = \bigoplus_{m \geq 0} R_m$ be a graded ring. Assume that R is a domain and R_0 is a field.

- (1) If the n -th Veronese subring $R^{(n)} := \bigoplus_{m \geq 0} R_{mn}$ is a finitely generated R_0 -algebra for some $n > 0$, then so is R .
- (2) Let $\mathfrak{a} \subseteq R$ be a nonzero homogeneous ideal, and suppose that R is a finitely generated R_0 -algebra. If $R^{(n)}/\mathfrak{a}^{(n)}$ is a finite dimensional R_0 -vector space for some $n > 0$, then so is R/\mathfrak{a} , where $\mathfrak{a}^{(n)} := \bigoplus_{m \geq 0} \mathfrak{a}_{mn}$.

Proof. For the proof of (1) we refer the proof of [HK10, Lemma 5.68]. For (2), let $l > 0$ be an integer divisible enough. Then there exists $n_0 > 0$ such that $\mathfrak{a}_{l+n} \subseteq R_{l+n} = R_n \cdot R_l = R_n \cdot \mathfrak{a}_l \subseteq \mathfrak{a}_{l+n}$ for each $n \geq n_0$, and hence $\mathfrak{a}_m = R_m$ for each $m \gg 0$, which is our claim. \square

As mentioned after Notation 3.7, the assumption (ii) of the main theorem (Theorem 1.1) satisfied if and only if $R/R_S(X_{\bar{\eta}}, \Delta_{\bar{\eta}}, a(K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}))$ is finite dimensional as k -vector space. This condition is equivalent to the condition that $R/R_S(X_{\bar{\eta}}, \Delta_{\bar{\eta}}, an(K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}))$ is finite dimensional for an integer $n > 0$ by (2) of the above lemma.

Definition 3.10. In the situation of Notation 3.1, we denote the kernel of $\phi_{(X, \Delta)}^{(de)} : F_X^{de} * \mathcal{L}_{(X, \Delta)}^{(de)} \rightarrow \mathcal{O}_X$ by $\mathcal{B}_{(X, \Delta)}^{de}$ for every integer $e > 0$. When $\Delta = 0$, we denote $\mathcal{B}_{(X, 0)}^e$ by \mathcal{B}_X^e .

Example 3.11. In the situation of Notation 3.1, assume that X is projective, and $(p^c - 1)(K_X + \Delta)$ is Cartier for some $c > 0$ divisible by d . Let H be an ample Cartier divisor. We show that $R/R_S(X, \Delta, H)$ is finite dimensional if and only if (X, Δ) is F -pure. By the Fujita vanishing theorem, there is an $m > 0$ such that $H^1(X, \mathcal{B}_{(X, \Delta)}^c(mH + N)) = 0$ for every nef Cartier divisor N . We may assume that $mH - (K_X + \Delta)$ is nef. If (X, Δ) is F -pure, or equivalently if $\phi_{(X, \Delta)}^{(c)}$ is surjective, then so is the morphism $H^0(X, \phi_{(X, \Delta)}^{(c)} \otimes \mathcal{O}_X(mH + N))$. Furthermore we see that $H^0(X, \phi_{(X, \Delta)}^{(ce)} \otimes \mathcal{O}_X(mH + N))$ is also surjective for each $e > 0$, because of the definition of $\phi_{(X, \Delta)}^{(ce)}$ and the following isomorphisms

$$\begin{aligned} & \left(F_X^{ce} (\phi_{(X, \Delta)}^{(c)} \otimes \mathcal{L}_{(X, \Delta)}^{(ce)}) \right) \otimes \mathcal{O}_X(mH + N) \\ & \cong F_X^{ce} \left(\phi_{(X, \Delta)}^{(c)} \otimes \mathcal{O}_X(mH + (p^{ce} - 1)(mH - (K_X + \Delta)) + p^{ce}N) \right). \end{aligned}$$

This implies that $S^0(X, \Delta, mH + N) = H^0(X, mH + N)$ and that $R/R_S(X, \Delta, H)$ is finite dimensional. Conversely it is clear that if $R/R_S(X, \Delta, H)$ is finite dimensional, then $\phi_{(X, \Delta)}^{(c)}$ is surjective, or equivalently, (X, Δ) is F -pure.

The above example shows that if $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is F -pure and $K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}$ is an ample $\mathbb{Z}_{(p)}$ -Cartier divisor, then the assumption (ii) (and (i)) of the main theorem (Theorem 1.1) holds. We next consider the value of such m_0 in the case when $X_{\bar{\eta}}$ a curve. Corollary 3.14 provides a value of such m_0 effectively when $K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}$ is ample.

Lemma 3.12. *Let X be a Gorenstein projective curve, and let H be an ample Cartier divisor such that $H - K_X$ is nef. Then for each integer $e, m \geq 1$,*

$$H^1(X, \mathcal{B}_X^e \otimes \mathcal{O}_X(K_X + mH)) = 0.$$

Moreover if X is F -pure, then

$$S^0(X, K_X + mH) = H^0(X, K_X + mH).$$

Proof. Clearly the second statement follows from the first and the long exact sequence of cohomology induced from the surjective morphism $\phi_{(X, \Delta)}^{(e)} \otimes \mathcal{O}_X(K_X + mH)$. We prove the first statement. Let $\nu : C \rightarrow X$ be the normalization. Then a commutative diagram of varieties

$$\begin{array}{ccc} C & \xrightarrow{F_C^e} & C \\ \nu \downarrow & & \downarrow \nu \\ X & \xrightarrow{F_X^e} & X \end{array}$$

induces a commutative diagram of \mathcal{O}_X -modules:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \nu_* \mathcal{B}_C^e(K_C) & \longrightarrow & \nu_* F_C^e \omega_C & \xrightarrow{\nu_* \text{Tr}_{F_C^e}} & \nu_* \omega_C \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow F_X^e \text{Tr}_{\nu} & & \downarrow \text{Tr}_{\nu} \\ 0 & \longrightarrow & \mathcal{B}_X^e(K_X) & \longrightarrow & F_X^e \omega_X & \xrightarrow{\text{Tr}_{F_X^e}} & \omega_X \end{array}$$

Since each vertical morphism is an isomorphism on some dense open subset of X , the kernel and the cokernel of α are torsion \mathcal{O}_X -modules. Furthermore since \mathcal{B}_C^e has no torsion, we see that α is injective. For each $m > 0$, the following exact sequence

$$0 \rightarrow (\nu_* \mathcal{B}_C^e(K_C))(mH) \xrightarrow{\alpha \otimes \mathcal{O}_X(mH)} \mathcal{B}_X^e(K_X + mH) \rightarrow \text{coker}(\alpha) \rightarrow 0,$$

induces a surjection

$$H^1(C, \mathcal{B}_C^e(K_C + m\nu^*H)) \cong H^1(X, (\nu_* \mathcal{B}_C^e(K_C))(mH)) \rightarrow H^1(X, \mathcal{B}_X^e(K_X + mH)).$$

Moreover, since ν^*H is ample and

$$\nu^*H - K_C = \nu^*(H - K_X) + \nu^*K_X - K_C \sim \nu^*(H - K_X) + E$$

is nef, where E is effective divisor on C defined by the conductor ideal, we may assume that X is smooth. Then we have $H^1(X, mpH) = H^0(X, K_X - mpH) = 0$ for each $m \geq 1$ by the Serre duality. For each $m \geq 1$ there exists an exact sequence

$$0 \rightarrow \mathcal{O}_X(mH) \rightarrow F_{X*} \mathcal{O}_X(mpH) \rightarrow \mathcal{B}_X^1(K_X + mH) \rightarrow 0$$

induced by Cartier operator, which shows that $H^1(X, \mathcal{B}_X^1(K_X + mH)) = 0$. This implies $H^0(X, \phi_X^{(1)} \otimes \mathcal{O}_X(K_X))$ is surjective, and thus $H^0(X, \phi_X^{(e)} \otimes \mathcal{O}_X(K_X + mH))$ is also surjective for every $e, m \geq 1$ because of the definition of $\phi_X^{(e)}$. Hence the exact sequence

$$0 \rightarrow \mathcal{B}_X^e(K_X + mH) \rightarrow F_{X*}^e \mathcal{O}_X(K_X + mp^eH) \xrightarrow{\phi_X^{(e)} \otimes \mathcal{O}_X(K_X + mH)} \mathcal{O}_X(K_X + mH) \rightarrow 0$$

induces the following:

$$H^1(X, \mathcal{B}_X^e(K_X + mH)) \hookrightarrow H^1(X, \mathcal{O}_X(K_X + mp^eH)) \cong H^0(X, -mp^eH) = 0.$$

□

Proposition 3.13. *In the situation of Notation 3.1, let X be a projective curve, let $K_X + \Delta$ is nef and let H be a Cartier divisor. Assume either that (i) $H + (a-1)K_X$ is ample and $H + (a-2)K_X$ is nef, or that (ii) $X \cong \mathbb{P}^1$ and H is ample. Then for each $e > 0$,*

$$H^1(X, \mathcal{B}_{(X,\Delta)}^{de} \otimes \mathcal{O}_X(a(K_X + \Delta) + H)) = 0.$$

Moreover if (X, Δ) is F -pure, then

$$S^0(X, \Delta, a(K_X + \Delta) + H) = H^0(X, a(K_X + \Delta) + H).$$

Proof. Clearly the second statement follows from the first and the long exact sequence of cohomology induced from the surjective morphism $\phi_{(X,\Delta)}^{(e)} \otimes \mathcal{O}_X(K_X + mH)$. We prove the first statement. Let E' be an effective Cartier divisor satisfying $\mathcal{O}_X(E') \subseteq \mathcal{O}_X(E)$ and $\Delta' := E'/a$. For each $e > 0$ there is a commutative diagram

$$\begin{array}{ccc} F_X^{de} \mathcal{L}_{(X,\Delta)}^{(de)}(p^{de}(a(K_X + \Delta) + H)) & \xrightarrow{\phi_{(X,\Delta)}^{(de)} \otimes \mathcal{O}_X(a(K_X + \Delta) + H)} & \mathcal{O}_X(a(K_X + \Delta) + H) \\ \uparrow & & \uparrow \\ F_X^{de} \mathcal{L}_{(X,\Delta')}^{(de)}(p^{de}(a(K_X + \Delta') + H)) & \xrightarrow{\phi_{(X,\Delta')}^{(de)} \otimes \mathcal{O}_X(a(K_X + \Delta') + H)} & \mathcal{O}_X(a(K_X + \Delta') + H) \end{array}$$

where the vertical morphisms are natural inclusion. This induces the injective morphism

$$\mathcal{B}_{(X, \Delta')}^{de}(a(K_X + \Delta') + H) \rightarrow \mathcal{B}_{(X, \Delta)}^{de}(a(K_X + \Delta) + H)$$

whose cokernel is a torsion \mathcal{O}_X -module. Hence it suffices to prove that $H^1(X, \mathcal{B}_{(X, \Delta)}^{de}(a(K_X + \Delta') + H)) = 0$. When (i) holds, we set $E' = 0$. By the previous lemma we have $H^1(X, \mathcal{B}_X^{de}(aK_X + H)) = 0$. When (ii) holds, we may assume $a(K_X + \Delta') \sim 0$. Then it is easily seen that $\dim H^1(X, \mathcal{B}_{(X, \Delta')}^{de}) \leq 1$. Since every vector bundle on \mathbb{P}^1 is isomorphic to a direct sum of line bundles, we have $H^1(X, \mathcal{B}_{(X, \Delta')}^{de}(H)) = 0$. This completes the proof. \square

The following corollary will be used to prove weak positivity theorem for fibrations of relative dimension one (Corollary 5.4).

Corollary 3.14. *In the situation of Notation 3.1, assume that X is a projective curve and (X, Δ) is F -pure. If $K_X + \Delta$ is ample (resp. K_X is ample and $a \geq 2$), then for each $m \geq 2$ (resp. $m \geq 1$),*

$$S^0(X, \Delta, am(K_X + \Delta)) = H^0(X, am(K_X + \Delta)).$$

Proof. We note that a Gorenstein curve has nef dualizing sheaf unless it is isomorphic to \mathbb{P}^1 . Hence the statement follows from the above proposition. \square

Remark 3.15. F -pure singularities of curves are completely classified [GW77]. For example, nodes are F -pure singularities, but cusps are not.

We next study Frobenius stable canonical rings of varieties with semi-ample canonical bundles in any dimension. For such varieties, Corollary 3.18 provides a criterion of the finiteness of the dimension of R/R_S in terms of the singularity of the canonical models. This is obtained as an application of Theorem 3.17, which is a kind of canonical bundle formula.

In order to formulate the problem, we start with an observation of Iitaka fibrations.

Observation 3.16. Let X be a normal projective variety, and let Δ be an effective $\mathbb{Z}_{(p)}$ -Weil divisor on X such that $K_X + \Delta$ is a semi-ample \mathbb{Q} -Cartier divisor. Let

$$f : X \rightarrow Y := \text{Proj } R(X, K_X + \Delta)$$

be the Iitaka fibration. Then there exists an ample \mathbb{Q} -Cartier divisor H on Y satisfying $f^*H \sim K_X + \Delta$. Let $Y_0 \subseteq Y$ be an open subset such that $f_0 := f|_{X_0} : X_0 \rightarrow Y_0$ is flat, where $X_0 := f^{-1}(Y_0)$.

(I) Assume that $R_S(X, \Delta, K_X + \Delta) \neq 0$. Then there exists an integer $m > 0$ such that $m\Delta$ is integral and $S^0(X, \Delta, m(K_X + \Delta)) \neq 0$. This implies that

$$S^0(X, \Delta, ((m-1)p^{e'} + 1)(K_X + \Delta)) \neq 0$$

for some $e' > 0$ divisible enough. Since $p \nmid (m-1)p^{e'} + 1$, there exists an $e > 0$ such that $S^0(X, \mathcal{O}_X((p^e - 1)(K_X + \Delta))) \neq 0$. We set $R' := (1 - p^e)(K_X + \Delta)$. Let η be the generic point of Y . By the assumption, $\mathcal{O}_X(-R')|_{X_\eta}$ is a torsion line bundle on X_η with nonzero global sections, and thus it is trivial. Hence $\mathcal{O}_X(R')|_{X_\eta}$ is also trivial, and $f_*\mathcal{O}_X(R')$ is a torsion free sheaf on Y of rank one. Then there exists an

effective Weil divisor B supported on $X \setminus X_0$ such that $f_*\mathcal{O}_X(R' + B) \cong \mathcal{O}_Y(S)$ for some Weil divisor S on Y . We set $R := R' + B = (1 - p^e)(K_X + \Delta) + B$. Then

$$R = K_X + \Delta + B - p^e(K_X + \Delta) \sim_{\mathbb{Q}} K_X + \Delta - p^e f^* H.$$

Replacing e , we may assume that $p^e H$ is $\mathbb{Z}_{(p)}$ -Cartier, and thus there exists an integer $a > 0$ not divisible by p such that $a\Delta$ is integral and $H' := ap^e H$ is Cartier. Then we have

$$aR \sim a(K_X + \Delta) + aB - f^* H'.$$

(II) In the situation of (I), after replacing e by its multiple, we assume that $(p^e - 1)(K_X + \Delta)$ is base point free. Then we may take $(p^e - 1)H$ as Cartier. In this case we have $\mathcal{O}_X(R') \cong f^*\mathcal{O}_Y((1 - p^e)H)$, and thus we may choose $B = 0$, $R = R'$ and $S = (1 - p^e)H$ by projection formula. In particular we have $R \sim f^* S$.

In a more general situation than the above, we prove the following theorem which is a kind of canonical bundle formula (see [DS15, Theorem B] for a related result).

Theorem 3.17. *Let $f : X \rightarrow Y$ be a fibration between normal varieties, let Δ be an effective \mathbb{Q} -Weil divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p , and let Y_0 be a smooth open subset of Y such that $\text{codim} Y \setminus Y_0 \geq 2$ and $f_0 := f|_{X_0} : X_0 \rightarrow Y_0$ is flat, where $X_0 := f^{-1}(Y_0)$. Further assume that the following conditions:*

- (i) $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is globally F -split where $\bar{\eta}$ is geometric generic point of Y .
- (ii) There exists a Weil divisor R on X , such that $f_*\mathcal{O}_X(R) \cong \mathcal{O}_Y(S)$ for some Weil divisor S on Y and $aR \sim a(K_X + \Delta) + B - f^*C$ for some effective Weil divisor B supported on $X \setminus X_0$ and for some Cartier divisor C on Y .

Then, there exists an effective \mathbb{Q} -Weil divisor Δ_Y on Y , which satisfies the following conditions:

- (1) $a'\Delta_Y$ is integral for some integer $a' > 0$ divisible by a but not by p , and $\mathcal{O}_Y(a'(K_Y + \Delta_Y - S)) \cong \mathcal{O}_Y(a'a^{-1}C) \cong f_*\mathcal{O}_X(a'(K_X + \Delta + a^{-1}B - R))$.
- (2) For every effective Weil divisor B' supported on $X \setminus X_0$ and for every Cartier divisor D on Y ,

$$S^0(X, \Delta, B' + f^*D + R) \cong S^0(Y, \Delta_Y, D + S).$$

- (3) If f is a birational morphism, then $\Delta_Y = f_*\Delta$.
- (4) Suppose that X_0 is Gorenstein and $R|_{X_0}$ is Cartier. Let Γ be an effective Cartier divisor on X_0 defined by the image of the natural morphism

$$\mathcal{O}_{X_0}(-R|_{X_0}) \otimes f_0^*(f_0_*\mathcal{O}_{X_0}(R|_{X_0})) \rightarrow \mathcal{O}_{X_0},$$

and let y be a point of Y_0 . Then the following conditions are equivalent:

- (a) $\text{Supp } \Delta$ does not contain any irreducible component of $f^{-1}(y)$, and $(X_{\bar{y}}, \Delta_{\bar{y}})$ is globally F -split, where \bar{y} is the algebraic closure of y .
- (b) y is not contained in $f(\text{Supp } \Gamma) \cup \text{Supp } \Delta_Y$.

Note that if R is linearly equivalent to the pullback of a Cartier divisor on Y , then replacing C , we may assume that $R = 0$, $S = 0$ and $\Gamma = 0$.

For varieties with semi-ample canonical bundles, Corollary 3.18 provides a criterion of the finiteness of the dimension of R/R_S in terms of the singularity of the

canonical models. As explained after Notation 3.7, the finiteness of R/R_S is equivalent to the assumption (ii) of the main theorem (Theorems 1.1 or 5.1). We remark that for such varieties, the assumption (i) of the main theorem, that is the finitely generation of canonical rings, is always satisfied.

Corollary 3.18. *In the situation of Observation 3.16 (I),*

- (1) $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is globally F -split, where $\bar{\eta}$ is the geometric generic point of Y . In particular, f is separable.
- (2) Let Δ_Y be as in Theorem 3.17. In the situation of Observation 3.16 (II) (i.e. $l(K_X + \Delta)$ is Cartier for an integer $l > 0$ not divisible by p), $R/R_S(X, \Delta, K_X + \Delta)$ is a finite dimensional k -vector space if and only if (Y, Δ_Y) is F -pure.

Before the proof of Theorem 3.17 and Corollary 3.18, we observe morphisms induced by the push-forward of the trace map of the relative Frobenius morphism. This observation will be also referred in the proof of Theorem 6.13.

Observation 3.19. Let $f : X \rightarrow Y$ be a projective morphism from a Gorenstein variety X to a smooth variety Y . Let $\Delta = E/a$ be an effective $\mathbb{Z}_{(p)}$ -AC divisor on X whose support does not contain any irreducible component of any fiber of f . Let d be the smallest positive integer satisfying $a|(p^d - 1)$. Let $e \geq 0$ be an integer.

(I) For every $y \in Y$, we have the following diagram:

$$\begin{array}{ccccc}
 (X_{\bar{y}})^{de} & \xlongequal{\quad} & X_{\bar{y}^{de}}^{de} & \longrightarrow & X^{de} \\
 & & F_{X_{\bar{y}}/Y}^{(de)} \downarrow & & \downarrow F_{X/Y}^{(de)} \\
 & & X_{\bar{y}^{de}} & \longrightarrow & X_{Y^{de}} \\
 & & \downarrow & & \downarrow f_{Y^{de}} \\
 & & \bar{y}^{de} & \longrightarrow & Y^{de}
 \end{array}$$

Let R be a Cartier divisor on X . We denote by $\theta^{(de)}$ the morphism

$$f_{Y^{de}*}(\phi_{(X, \Delta)/Y}^{(de)} \otimes \mathcal{O}_X(R)_{Y^{de}}) : f^{(de)}_* \mathcal{L}_{(X, \Delta)/Y}^{(de)}(p^{de}R) \rightarrow f_{Y^{de}*} \mathcal{O}_{X_{Y^{de}}}(R)_{Y^{de}}.$$

Here we recall that $\mathcal{L}_{(X, \Delta)/Y}^{(de)} := \mathcal{O}_{X^{de}}((1 - p^{de})(K_{X^{de}/Y^{de}} + \Delta))$.

(II) Let $Y_0 \subseteq Y$ be an open subset such that $f_0 := f|_{X_0} : X_0 \rightarrow Y_0$ is flat, where $X_0 := f^{-1}(Y_0)$. Assume that $y \in Y_0$ and that $E|_{Y_0}$ is Cartier. Since f_0 is a Gorenstein morphism, $X_{\bar{y}}$ is Gorenstein. Set $\Delta_{\bar{y}} = E|_{X_{\bar{y}}}/a$. Then $\mathcal{L}_{(X, \Delta)/Y}^{(de)}|_{(X_{\bar{y}})^{de}} \cong \mathcal{L}_{(X_{\bar{y}}/\bar{y}, \Delta_{\bar{y}})}^{(de)}$ and we have the following diagram of $k(\bar{y}^{de})$ -vector spaces for every $e > 0$:

$$\begin{array}{ccc}
 H^0((X_{\bar{y}})^{de}, \mathcal{L}_{(X_{\bar{y}}/\bar{y}, \Delta_{\bar{y}})}^{(de)} \otimes \mathcal{O}_X(p^{de}R)|_{(X_{\bar{y}})^{de}}) & \longleftarrow & \left(f^{(de)}_* \mathcal{L}_{(X, \Delta)/Y}^{(de)}(p^{de}R) \right) \otimes k(\bar{y}^{de}) \\
 \downarrow & & \downarrow \theta^{(de)} \otimes k(\bar{y}^{de}) \\
 H^0(X_{\bar{y}^{de}}, \phi_{(X_{\bar{y}}/\bar{y}, \Delta_{\bar{y}})}^{(de)} \otimes \mathcal{O}_X(p^{de}R)|_{X_{\bar{y}^{de}}}) & & \\
 \downarrow & & \downarrow \\
 H^0(X_{\bar{y}^{de}}, \mathcal{O}_X(R)|_{X_{\bar{y}^{de}}}) & \longleftarrow & (f_{Y^{de}*} \mathcal{O}_X(R)_{Y^{de}}) \otimes k(\bar{y}^{de})
 \end{array}$$

(III) Let $Y_1 \subseteq Y_0$ be an open subset such that $\dim H^0(X_y, \mathcal{O}_X(R)|_{X_y})$ is a constant function on Y_1 with value h . If $y \in Y_1$, then the horizontal morphisms in the above diagram are isomorphisms by [Har77, Corollary 12.9]. Hence for every $e > 0$ we have

$$\begin{aligned} & \dim_{k(\bar{y})} \text{im}(H^0(X_{\bar{y}}, \phi_{(X_{\bar{y}}, \Delta_{\bar{y}})}^{(de)} \otimes \mathcal{O}_X(R)|_{X_{\bar{y}}})) \\ &= \dim_{k(\bar{y}^{de})} \text{im}(H^0(X_{\bar{y}^{de}}, \phi_{(X_{\bar{y}}/\bar{y}, \Delta_{\bar{y}})}^{(de)} \otimes \mathcal{O}_X(R)|_{X_{\bar{y}^{de}}})) \\ &= \dim_{k(\bar{y}^{de})} \text{im}(\theta^{(de)} \otimes k(\bar{y}^{de})) \\ &= h - \dim_{k(\bar{y}^{de})} \text{coker}(\theta^{(de)} \otimes k(\bar{y}^{de})) \\ &= h - \dim_{k(\bar{y}^{de})} (\text{coker}(\theta^{(de)})) \otimes k(\bar{y}^{de}). \end{aligned}$$

Here, the last equality follows from the right exactness of the tensor functor.

(IV) Assume that $(p^d - 1)(K_{X/Y} + \Delta - R)|_{X_1} \sim f_1^*C$ for some Cartier divisor C on Y_1 , where $X_1 := f^{-1}(Y_1)$ and $f_1 := f|_{X_1} : X_1 \rightarrow Y_1$. Then

$$\mathcal{L}_{(X_{\bar{y}}, \Delta_{\bar{y}})}^{(de)} \otimes \mathcal{O}_{X^{de}}(p^{de}R)|_{(X_{\bar{y}})^{de}} \cong \mathcal{O}_{X^{de}}(R)|_{(X_{\bar{y}})^{de}}$$

for every $y \in Y_1$. Thus we can regard $H^0(X_{\bar{y}}, \phi_{(X_{\bar{y}}, \Delta_{\bar{y}})}^{(de)} \otimes \mathcal{O}_X(R)|_{X_{\bar{y}}})$ as the e -th iteration of the $(p^{-d}$ -linear) morphism

$$\tau := H^0(X_{\bar{y}}, \phi_{(X_{\bar{y}}, \Delta_{\bar{y}})}^{(d)} \otimes \mathcal{O}_X(R)|_{X_{\bar{y}}}) : H^0(X_{\bar{y}}, \mathcal{O}_X(R)|_{X_{\bar{y}}}) \rightarrow H^0(X_{\bar{y}}, \mathcal{O}_X(R)|_{X_{\bar{y}}}).$$

If $e \geq h$, then $\text{im}(\tau^e) = \text{im}(\tau^h)$, and thus

$$\text{im}(H^0(X_{\bar{y}}, \phi_{(X_{\bar{y}}, \Delta_{\bar{y}})}^{(de)} \otimes \mathcal{O}_X(R)|_{X_{\bar{y}}})) = S^0(X_{\bar{y}}, \Delta_{\bar{y}}, \mathcal{O}_X(R)|_{X_{\bar{y}}}).$$

Hence by (3), we see that

$$\dim_{k(\bar{y})} S^0(X_{\bar{y}}, \Delta_{\bar{y}}, \mathcal{O}_X(R)|_{X_{\bar{y}}}) = h - \dim_{k(\bar{y}^{de})} (\text{coker}(\theta^{(de)})) \otimes k(\bar{y}^{de}).$$

In particular, since the function $\dim_{k(\bar{y}^{de})} (\text{coker}(\theta^{(de)})) \otimes k(\bar{y}^{de})$ on Y^{de} is upper semicontinuous, the function $\dim_{k(\bar{y})} S^0(X_{\bar{y}}, \Delta_{\bar{y}}, \mathcal{O}_X(R)|_{X_{\bar{y}}})$ on Y_1 is lower semicontinuous.

Proof of Theorem 3.17. Let $d > 0$ be an integer such that $a|(p^d - 1)$.

Step1. We define Δ_Y and we show that this is independent of the choice of d . We first note that, for each $e \geq 0$ there exist isomorphisms

$$\begin{aligned} & f_* \mathcal{O}_X((1 - p^{de})(K_X + \Delta) + p^{de}R) \\ & \cong f_* \mathcal{O}_X((1 - p^{de})(K_X + \Delta + a^{-1}B - R) + (p^{de} - 1)a^{-1}B + R) \\ & \cong \mathcal{O}_Y((1 - p^{de})a^{-1}C) \otimes f_* \mathcal{O}_X((p^{de} - 1)a^{-1}B + R) \\ & \cong \mathcal{O}_Y((1 - p^{de})a^{-1}C) \otimes f_* \mathcal{O}_X(R) \\ & \cong \mathcal{O}_Y((1 - p^{de})a^{-1}C + S). \end{aligned}$$

Since Y is normal, to define Δ_Y we may assume $Y = Y_0$ and X is smooth. Then for each $e > 0$ we have

$$\begin{aligned} & f_*^{(de)} \mathcal{L}_{(X, \Delta)/Y}^{(de)}(p^{de}R) \cong \mathcal{O}_{Y^{de}}((1 - p^{de})(a^{-1}C - K_{Y^{de}}) + S) \quad \text{and} \\ & f_{Y^{de}*} \mathcal{O}_{X^{de}}(R_{Y^{de}}) \cong F_Y^{de*} f_* \mathcal{O}_X(R) \cong \mathcal{O}_{Y^{de}}(p^{de}S), \end{aligned}$$

thus

$$\theta^{(de)} := f_{Y^{de}*} \phi_{(X, \Delta)/Y}^{(de)} \otimes \mathcal{O}_{X_{Y^{de}}}(R_{Y^{de}}) : f_*^{(de)} \mathcal{L}_{(X, \Delta)/Y}^{(de)}(p^{de}R) \rightarrow f_{Y^{de}*} \mathcal{O}_{X_{Y^{de}}}(R_{Y^{de}})$$

is a homomorphism between line bundles. By the assumption of the global F -splitting of $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$, we see that the left vertical morphism of the diagram in Observation 3.19 (II) (for $y = \eta$) is surjective, and hence by Observation 3.19 (III) $\theta^{(de)}$ is generically surjective for every $e > 0$. Thus $\theta^{(de)}$ defines an effective Cartier divisor $E^{(de)}$ on Y . Then for every $e > 1$ we have $E^{(de)} = p^d E^{(d(e-1))} + E^{(d)}$, because relations between morphisms

$$\begin{aligned} \theta^{(de)} &:= f_{Y^{de}*} \phi_{(X, \Delta)/Y}^{(de)} \otimes \mathcal{O}_{X_{Y^{de}}}(R_{Y^{de}}) \\ &= f_{Y^{de}*} \left(\phi_{(X, \Delta)/Y}^{(d(e-1))} \otimes \mathcal{O}_{X_{Y^{d(e-1)}}}(R_{Y^{d(e-1)}}) \right)_{Y^{de}} \\ &\quad \circ f^{(d(e-1))} \circ f_{Y^{d(e-1)}*} \phi_{(X^{d(e-1)}, \Delta)/Y^{d(e-1)}}^{(d)} \otimes \left(\mathcal{L}_{(X, \Delta)/Y}^{(d(e-1))} \right)_{Y^{de}} (p^{d(e-1)} R_{Y^{de}}) \\ &\cong (F_Y^{d*} \theta^{(d(e-1))}) \circ (\theta^{(d)} \otimes \mathcal{O}_{Y^{de}}((p^{d(e-1)} - 1)(a^{-1}C - K_{Y^{de}}))) \end{aligned}$$

implies that $E^{(de)} = (p^{d(e-1)} + \dots + p + 1)E^{(d)} = (p^{de} - 1)(p^d - 1)^{-1}E^{(d)}$ for every $e > 0$. We define $\Delta_Y := (p^d - 1)^{-1}E^{(d)}$, this is independent of the choice of d by the above. Note that by this definition

$$\begin{aligned} f_* \mathcal{O}_X((p^d - 1)(K_{X/Y} + \Delta - R)) &\cong \mathcal{O}_Y((p^d - 1)(a^{-1}C - K_Y)) \\ &\cong \mathcal{O}_Y((p^d - 1)(\Delta_Y - S)), \end{aligned}$$

which proves (1).

Step2. We show that for each $e > 0$ there exists a commutative diagram

$$\begin{array}{ccc} F_Y^{de*} \mathcal{L}_{(Y, \Delta_Y)}^{(de)} & \xrightarrow{(\phi_{(Y, \Delta_Y)}^{(de)} \otimes \mathcal{O}_Y(S))^{**}} & \mathcal{O}_Y(S) \\ \cong \downarrow & & \downarrow \cong \\ f_* F_X^{de*} \mathcal{L}_{(X, \Delta)}^{(de)}(p^{de}R) & \xrightarrow{\psi^{(de)}} & f_* \mathcal{O}_X(R) \end{array}$$

where $\psi^{(de)} := f_*((\phi_{(X, \Delta)}^{(de)} \otimes \mathcal{O}_X(R))^{**})$. It is clear that each object of the above diagram is a reflexive sheaf, so we may assume that $Y = Y_0$ and X is smooth. Since $F_X^d = (F_Y^d)_X \circ F_{X/Y}^{(d)}$, we have

$$\begin{aligned} \phi_{(X, \Delta)}^{(d)} \otimes \mathcal{O}_X(R) &:= \text{Tr}_{F_X^d} \otimes \omega_X^{-1}(R) \cong \left(\text{Tr}_{(F_Y^d)_X} \circ (F_Y^d)_X \text{Tr}_{F_{X/Y}^{(d)}} \right) \otimes \omega_X^{-1}(R) \\ &\cong \left(((f^* \text{Tr}_{F_Y}) \otimes \omega_{X/Y}) \circ (F_Y^d)_X \text{Tr}_{F_{X/Y}^{(d)}} \right) \otimes \omega_X^{-1}(R) \\ &\cong (f^* \phi_Y^{(d)} \otimes \mathcal{O}_X(R)) \circ (F_Y^d)_X \left(\phi_{(X, \Delta)/Y}^{(d)} \otimes f_{Y^d*} \omega_{Y^d}^{1-p^d}(R_{Y^d}) \right). \end{aligned}$$

We note that $\phi_Y^{(d)}$ is a morphism between vector bundles on Y , thus $\psi^{(d)}$ is decomposed into

$$\psi^{(d)} \cong (\phi_Y^{(d)} \otimes \mathcal{O}_Y(S)) \circ F_{Y^d*}(\theta^{(d)} \otimes \omega_{Y^d}^{1-p^d}).$$

On the other hand, by the definition of Δ_Y , there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y((1-p^d)\Delta_Y + p^dS) & \longrightarrow & \mathcal{O}_{Y^d}(p^dS) \\ \cong \downarrow & & \downarrow \cong \\ f^{(d)}_* \mathcal{L}_{(X,\Delta)/Y}^{(d)}(p^dR) & \xrightarrow{\theta^{(d)}} & f_{Y^d*} \mathcal{O}_{X_{Y^d}}(R_{Y^d}). \end{array}$$

Applying the functor $F_{Y*}^d((_) \otimes \omega_Y^{1-p^d})$ to this diagram, we have the following:

$$\begin{array}{ccc} (F_{Y*}^d \mathcal{O}_Y((1-p^d)(K_Y + \Delta_Y))) \otimes \mathcal{O}_Y(S) & \longrightarrow & (F_{Y*}^d \omega_Y^{1-p^d}) \otimes \mathcal{O}_Y(S) \\ \cong \downarrow & & \downarrow \cong \\ F_{Y*}^d f^{(d)}_* \mathcal{L}_{(X,\Delta)}^{(d)}(p^dR) & \xrightarrow{F_{Y*}^d(\theta^{(d)} \otimes \omega_Y^{1-p^d})} & (F_{Y*}^d \omega_Y^{1-p^d}) \otimes f_* \mathcal{O}_X(R). \end{array}$$

Hence by the decomposition of $\psi^{(d)}$ and the definition of $\phi_{(Y,\Delta_Y)}^{(d)}$, the claim is proved in the case when $e = 1$. Furthermore, for each $e > 0$ we have

$$\begin{aligned} \psi^{(d(e+1))} &\cong (f_*(\phi_{(X,\Delta)}^{(de)} \otimes \mathcal{O}_X(R))) \circ f_* F_X^{de}(\phi_{(X^{de},\Delta)}^{(d)} \otimes \mathcal{L}_{(X,\Delta)}^{(de)}(p^{de}R)) \\ &\cong \psi^{(de)} \circ F_Y^{de} f^{(de)}_*(\phi_{(X^{de},\Delta)}^{(d)} \otimes \mathcal{O}_{X^{de}}(R + (1-p^{de})(K_{X^{de}} + \Delta - R))) \\ &\cong \phi_{(Y,\Delta_Y)}^{(de)} \otimes \mathcal{O}_Y(S) \circ F_Y^{de}(\phi_{(Y^{de},\Delta_Y)}^{(d)} \otimes \mathcal{O}_{Y^{de}}(S + (1-p^{de})a^{-1}C)) \\ &\cong \phi_{(Y,\Delta_Y)}^{(de)} \otimes \mathcal{O}_Y(S) \circ F_Y^{de}(\phi_{(Y^{de},\Delta_Y)}^{(d)} \otimes \mathcal{O}_{Y^{de}}((1-p^{de})(K_Y + \Delta_Y) + p^{de}S)) \\ &\cong \phi_{(Y,\Delta_Y)}^{(de)} \otimes \mathcal{O}_Y(S) \circ F_Y^{de}(\phi_{(Y^{de},\Delta_Y)}^{(d)} \otimes \mathcal{L}_{(Y,\Delta_Y)}^{(de)}(p^{de}S)) \cong \phi_{(Y,\Delta_Y)}^{(d(e+1))} \otimes \mathcal{O}_Y(S). \end{aligned}$$

This is our claim.

Step3. We prove statements (2)-(4). (3) is obvious. We show (2). By the definition of $\phi_{(X,\Delta)}^{(d)}$, we may assume that X is smooth. Then there is a commutative diagram

$$\begin{array}{ccc} (F_Y^{de} f^{(de)}_* \mathcal{L}_{(X,\Delta)}^{(de)}(p^{de}R))(D) & \xrightarrow{\psi^{(de)} \otimes \mathcal{O}_Y(D)} & (f_* \mathcal{O}_X(R))(D) \\ \cong \downarrow & & \downarrow \cong \\ F_Y^{de} f^{(de)}_* \mathcal{L}_{(X,\Delta)}^{(de)}(p^{de}(f^*D + R)) & \xrightarrow{f_*(\phi_{(X,\Delta)}^{(de)} \otimes \mathcal{O}_Y(f^*D + R))} & f_* \mathcal{O}_X(f^*D + R) \\ \cong \downarrow & & \downarrow \cong \\ F_Y^{de} f^{(de)}_* \mathcal{L}_{(X,\Delta)}^{(de)}(p^{de}(f^*D + B' + R)) & \xrightarrow{f_*(\phi_{(X,\Delta)}^{(de)} \otimes \mathcal{O}_X(f^*D + B' + R))} & f_* \mathcal{O}_X(f^*D + B' + R). \end{array}$$

Thus by Step2,

$$\begin{aligned} H^0(X, \phi_{(X,\Delta)}^{(de)} \otimes \mathcal{O}_X(f^*D + B' + R)) &\cong H^0(Y, f_*(\phi_{(X,\Delta)}^{(de)} \otimes \mathcal{O}_X(f^*D + B' + R))) \\ &\cong H^0(Y, (\phi_{(Y,\Delta_Y)}^{(de)} \otimes (D + S))^{**}), \end{aligned}$$

which implies (2). For (4), we may assume that $Y = Y_0$. Then, since $f_* \mathcal{O}_X(R)$ is a line bundle, we only need to show that the case when $f_* \mathcal{O}_X(R) \cong \mathcal{O}_Y$ and

$R = \Gamma \geq 0$. In this case, since R and $(p^d - 1)(K_X + \Delta)$ are Cartier, and since f is flat projective, we have $H^0(X_y, \mathcal{O}_X(R)|_{X_y}) \neq 0$ and

$$H^0(X_y, \mathcal{O}_X((1 - p^d)R)|_{X_y}) = H^0(X_y, \mathcal{O}_{X_y}((1 - p^d)(K_{X_y} + \Delta_y))) \neq 0$$

for every $y \in Y$ by assumptions and upper semicontinuity [Har77, Theorem 12.8]. In particular, if X_y is reduced then $\mathcal{O}_X(R)|_{X_y} \cong \mathcal{O}_{X_y}$, because every nonzero endomorphism of a line bundle on a connected reduced projective scheme over a field is an isomorphism. Hence the isomorphism $\mathcal{O}_Y \cong f_* \mathcal{O}_X(R)$ shows that the support of R is contained a union of nonreduced fibers. Set $Y_1 := \{y \in Y \mid H^0(X_y, \mathcal{O}_X(R)_y) \cong k(y)\}$. Then we have

$$\text{Supp } \Delta_Y|_{Y_1} = \text{Supp } \text{coker}(\theta^{(d)})|_{Y_1} = \{y \in Y_1 \mid S^0(X_{\bar{y}}, \Delta_{\bar{y}}, \mathcal{O}_X(R)|_{X_{\bar{y}}}) = 0\},$$

where the first (resp. the second) equality follows from the definition of Δ_Y (resp. Observation 3.19 (IV)). Now we prove (a) \Rightarrow (b). In the situation of (a) X_y is reduced, and so $y \in Y \setminus f(\text{Supp } R)$. We recall Example 3.4, which shows that the global F -splitting of $(X_{\bar{y}}, \Delta_{\bar{y}})$ is equivalent to the equality $S^0(X_{\bar{y}}, \Delta_{\bar{y}}, \mathcal{O}_{X_{\bar{y}}}) = H^0(X_{\bar{y}}, \mathcal{O}_{X_{\bar{y}}})$. Thus it is enough to show that $y \in Y_1$. Let $\overline{\{y\}}$ be the closure in Y of the set $\{y\}$ with the reduced induced subscheme structure. Let Y' be a smooth open subset of $\overline{\{y\}}$ such that $R_{Y'} = 0$ and that $\text{Supp } \Delta$ does not contain any irreducible component of any fiber over Y' . Then for a general closed point $y' \in Y'$,

$$\begin{aligned} \dim_k S^0(X_{y'}, \Delta_{y'}, \mathcal{O}_{X_{y'}}) &= \dim_{k(\bar{y})} S^0(X_{\bar{y}}, \Delta_{y'}, \mathcal{O}_{X_{\bar{y}}}) \\ &= \dim_{k(\bar{y})} H^0(X_{\bar{y}}, \mathcal{O}_{X_{\bar{y}}}) = \dim_k H^0(X_{y'}, \mathcal{O}_{X_{y'}}), \end{aligned}$$

where the first (resp. the third) equality follows from lower semicontinuity proved in Observation 3.19 (IV) (resp. upper semicontinuity). Thus $(X_{y'}, \Delta_{y'})$ is globally F -split, and in particular $X_{y'}$ is reduced. Since k is algebraically closed, we have $H^0(X_{y'}, \mathcal{O}_{X_{y'}}) \cong k$, and hence $H^0(X_y, \mathcal{O}_{X_y}) \cong k(y)$, or equivalently, $y \in Y_1$. To prove (b) \Rightarrow (a), we replace Y by its affine open subset contained in $Y \setminus (\text{Supp } \Delta_Y \cup f(\text{Supp } R))$. Then the surjectivity of $\theta^{(d)}$ shows that $\phi_{(X, \Delta)/Y}^{(d)} : F_{X/Y}^{(d)} \mathcal{L}_{(X, \Delta)/Y}^{(d)} \rightarrow \mathcal{O}_{X_{Y^d}}$ is split, and thus so is $\phi_{(X, \Delta)/Y}^{(d)}|_{X_{\bar{y}^d}} : F_{X_{\bar{y}}/Y}^{(d)} (\mathcal{L}_{(X, \Delta)/Y}^{(d)}|_{(X_{\bar{y}})^d}) \rightarrow \mathcal{O}_{X_{\bar{y}^d}}$. This means that Δ does not contain any irreducible component of $f^{-1}(y)$, so $\Delta_{\bar{y}}$ is well-defined, and we have $\phi_{(X, \Delta)/Y}^{(d)}|_{X_{\bar{y}^d}} \cong \phi_{(X_{\bar{y}}, \Delta_{\bar{y}})/\bar{y}}^{(d)}$, which completes the proof. \square

Proof of Corollary 3.18. We use the notation of Observation 3.16. Let $l > 0$ be an integer such that $l(K_X + \Delta)$ is Cartier and base point free. We replace Y by its smooth locus Y_{sm} , and X by the smooth locus of $f^{-1}(Y_{\text{sm}})$. As in the proof of Theorem 3.17, we set

$$\psi^{(e)} := f_*(\phi_{(X, \Delta)}^{(e)} \otimes \mathcal{O}_X(R)) \text{ and } \theta^{(e)} := f_{Y^e*}(\phi_{(X, \Delta)/Y}^{(e)} \otimes \mathcal{O}_X(R)|_{Y^e})$$

for every $e > 0$ divisible enough. Since $S^0(X, \Delta, (p^d - 1)(K_X + \Delta)) \neq 0$, we have

$$\begin{aligned} 0 &\neq S^0(X, \Delta, (l - 1)(p^d - 1)(K_X + \Delta)) \\ &\hookrightarrow S^0(X, \Delta, (l - 1)(p^d - 1)(K_X + \Delta) + B) = S^0(X, \Delta, f^*l(p^d - 1)H + R). \end{aligned}$$

This implies the morphism

$$f_*(\phi_{(X, \Delta)}^{(e)} \otimes \mathcal{O}_X(f^*l(p^d - 1)H + R)) \cong \psi^{(e)} \otimes \mathcal{O}_Y(l(p^d - 1)H)$$

is nonzero for an $e > 0$ divisible enough, where the isomorphism follows from projection formula. Hence $\psi^{(e)}$ is also nonzero. By an argument similar to Step2, we can factor $\psi^{(e)}$ into $(\phi_Y^{(e)} \otimes \mathcal{O}_Y(S)) \circ F_{Y*}^e(\theta^{(e)} \otimes \omega_Y^{1-p^e})$, and hence $\theta^{(e)}$ is nonzero. Thus $\theta^{(e)} \otimes k(\bar{\eta}) \cong H^0(X_{\bar{\eta}}, \phi_{(X_{\bar{\eta}/\bar{\eta}}, \Delta_{\bar{\eta}})}^{(e)})$ is nonzero, or equivalently, $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is globally F -split. In particular $X_{\bar{\eta}}$ is reduced, and this means that f is separable. We show (2). First note that $R/R_S(X, \Delta, K_X + \Delta)$ is finite dimensional if and only if so is $R/R_S(X, \Delta, l(K_X + \Delta))$ by Lemma 3.9. Let $m \geq 0$ be an integer with $l|m$. Then we have $H^0(X, lm(K_X + \Delta)) \cong H^0(Y, lm(K_Y + \Delta))$. Furthermore, by Theorem 3.17 (2), we have

$$\begin{aligned} S^0(X, \Delta, m(K_X + \Delta)) &= S^0(X, \Delta, f^*(mH - S) + R) \\ &= S^0(Y, \Delta_Y, mH - S + S) = S^0(Y, \Delta_Y, mH). \end{aligned}$$

Thus $R/R_S(X, \Delta, l(K_X + \Delta)) \cong R/R_S(Y, \Delta_Y, lH)$. By Example 3.11, this k -vector space is finite dimensional if and only if (Y, Δ_Y) is F -pure, which is our claim. \square

Example 3.20. Let $f : X \rightarrow Y$ be a relatively minimal elliptic fibration. In other words, let f be a generically smooth fibration from a smooth projective surface X to a smooth projective curve Y , whose fibers have arithmetic genus one and do not contain (-1) -curves of X . Then by the canonical bundle formula [BM77, Theorem 2], we have

$$K_X \sim f^*D + \sum_{i=1}^r l_i F_i,$$

where D is a divisor on Y , $m_i F_i = X_{y_i}$ is a multiple fiber with the multiplicity m_i , and $0 \leq l_i < m_i$. Let m be the least common multiple of m_1, \dots, m_r , and let $a, e \geq 0$ be integers such that $m = ap^e$ and $p \nmid a$. We set

$$R := \sum_{i=1}^r \left\{ \frac{(1-p^d)l_i}{m_i} \right\} m_i F_i$$

for some $d \geq e$ satisfying $a|(p^d - 1)$. Here, recall that for every $s \in \mathbb{Q}$, $\{s\}$ is the fractional part $s - \lfloor s \rfloor$ of s . It is easily seen that $f_*\mathcal{O}_X(R) \cong \mathcal{O}_Y$ and

$$\begin{aligned} aK_X - aR &\sim a f^*D + a \sum_{i=1}^r l_i F_i - a \sum_{i=1}^r (1-p^d)l_i F_i + a \sum_{i=1}^r \left\lfloor \frac{(1-p^d)l_i}{m_i} \right\rfloor m_i F_i \\ &= f^*(aD + \sum_{i=1}^r \left(\frac{al_i p^d}{m_i} + a \left\lfloor \frac{(1-p^d)l_i}{m_i} \right\rfloor \right) y_i). \end{aligned}$$

Thus, a and R satisfy condition (ii) of Theorem 3.17. Furthermore, assume that the geometric generic fiber of f is globally F -split, or equivalently, is an elliptic curve with nonzero Hasse invariant. Then by Theorem 3.17 there exists an effective $\mathbb{Z}_{(p)}$ -divisor Δ_Y on Y such that

$$S^0(X, f^*D' + R) = S^0(Y, \Delta_Y, D')$$

for every divisor D' on Y , and $y_1, \dots, y_r \in f(R) \cup \Delta_Y$. Remark that if $p \nmid m_i$ for each i , then $m_i|(p^d - 1)$, and so $R = 0$.

Finally, applying Theorem 3.17, we show that for a smooth projective surface X of general type, $R/R_S(X, K_X)$ is finite dimensional (Corollary 3.23).

Corollary 3.21. *Let $f : X \rightarrow Y$ be a birational morphism between normal projective varieties, let Δ be an effective \mathbb{Q} -Weil divisor on X , and let $\Delta_Y := f_*\Delta$. Assume that $a(K_Y + \Delta_Y)$ is Cartier for some $a > 0$ not divisible by p and (Y, Δ_Y) is canonical. Then for each $m > 0$,*

$$S^0(Y, \Delta_Y, am(K_Y + \Delta_Y)) \cong S^0(X, \Delta, am(K_X + \Delta)).$$

Proof. Since (Y, Δ_Y) is canonical, $R := a(K_X + \Delta) - f^*a(K_Y + \Delta_Y)$ is an effective Weil divisor on X supported on the exceptional locus of f . Note that $f_*\mathcal{O}_X(R) \cong \mathcal{O}_Y$. We set $B := (a - 1)R$ and $B'_m := (m - 1)R$ for each $m \geq 1$. Then we have

$$aR = R + (a - 1)R = a(K_X + \Delta) - f^*a(K_Y + \Delta_Y) + B.$$

Thus, by Theorem 3.17, we have

$$\begin{aligned} S^0(Y, \Delta_Y, am(K_Y + \Delta_Y)) &\cong S^0(X, \Delta, B'_m + f^*am(K_Y + \Delta_Y) + R) \\ &\cong S^0(X, \Delta, am(K_X + \Delta)). \end{aligned}$$

□

Corollary 3.22 ([PST14, Exercise 5.15]). *Let $\varphi : Y \dashrightarrow Y'$ be a birational map between normal projective varieties, let Δ be an effective \mathbb{Q} -Weil divisor on Y , and let $\Delta' := \varphi_*\Delta$. Assume that $a(K_Y + \Delta)$ and $a(K_{Y'} + \Delta')$ are Cartier for some $a > 0$ not divisible by p , and that (Y, Δ) and (Y', Δ') are canonical. Then, for each $m > 0$,*

$$S^0(Y, \Delta, am(K_Y + \Delta)) \cong S^0(K_{Y'}, \Delta', am(K_{Y'} + \Delta')).$$

Proof. This follows directly from Corollary 3.21. □

The following corollary will be used to prove weak positivity theorem when geometric generic fibers are normal projective surfaces of general type with rational double point singularities (Corollary 5.5). Recall that the finiteness of the dimension of R/R_S is equivalent to the assumption (ii) of the main theorem (Theorems 1.1 or 5.1).

Corollary 3.23. *Let X be a normal projective surface of general type with rational double point singularities. If $p \geq 7$, then $R/R_S(X, K_X)$ is a finite dimensional vector space.*

Proof. By Corollary 3.22, we may assume that X is a smooth projective surface of general type which has no (-1) -curve. Then for each $n \gg 0$, nK_X is base point free, and $Y := \text{Proj } R(X, K_X)$ has only rational double point singularities [Băd01, Theorem 9.1]. When $p \geq 7$, Y is F -pure, because of the classification of rational double points [Art77, Section 3], and of Fedder's criterion [Fed83]. Hence the statement follows from Corollary 3.18. □

4. WEAK POSITIVITY AND NUMERICAL INVARIANT

In this section, we define an invariant of coherent sheaves on normal varieties which measures positivity. This will play an important role in the proof of the main theorem.

We first recall some definitions.

Definition 4.1. A coherent sheaf \mathcal{G} on a variety Y is said to be *generically globally generated* if the natural morphism $H^0(Y, \mathcal{G}) \otimes_k \mathcal{O}_Y \rightarrow \mathcal{G}$ is surjective on the generic point of Y .

Viehweg introduced the notion of weak positivity as a generalization of nefness of vector bundles.

Definition 4.2 ([Kol87, Notation, (vii)]). A coherent sheaf \mathcal{G} on a normal projective variety Y is said to be *weakly positive* if for every ample divisor H and for each $\alpha > 0$ there exist some $\beta > 0$ such that $(S^{\alpha\beta}\mathcal{G})^{**} \otimes \mathcal{O}_Y(\beta H)$ is generically globally generated. Here $(S^{\alpha\beta}\mathcal{G})^{**}$ is the double dual of the $\alpha\beta$ -th symmetric product of \mathcal{G} .

Remark 4.3. (1) The above definition is independent of the choice of H (e.g., [Vie95, Lemma 2.14]).

(2) \mathcal{G} is weakly positive if and only if its double dual \mathcal{G}^{**} is weakly positive.

(3) Assume that \mathcal{G} is a vector bundle on a smooth projective curve. Then \mathcal{G} is weakly positive if and only if it is nef.

Definition 4.4. Let Y be a normal variety, let \mathcal{G} be a coherent sheaf, and let H be an ample Cartier divisor. Then we define

$$T(Y, \mathcal{G}, H) := \left\{ \varepsilon \in \mathbb{Q} \left| \begin{array}{l} \text{there exists an integer } e > 0 \text{ such that} \\ p^e \varepsilon \in \mathbb{Z} \text{ and } (F_Y^{e*}\mathcal{G})(-p^e \varepsilon H) \text{ is} \\ \text{generically globally generated.} \end{array} \right. \right\},$$

$$t(Y, \mathcal{G}, H) := \sup T(Y, \mathcal{G}, H) \in \mathbb{R} \cup \{\infty\}.$$

Lemma 4.5. Let Y, \mathcal{G}, H be as in Definition 4.4, and let \mathcal{F} be a coherent sheaf.

- (1) If there exists a generically surjective morphism $\mathcal{F} \rightarrow \mathcal{G}$, then $t(Y, \mathcal{F}, H) \leq t(Y, \mathcal{G}, H)$.
- (2) $t(Y, \mathcal{F}, H) + t(Y, \mathcal{G}, H) \leq t(Y, \mathcal{F} \otimes \mathcal{G}, H)$.
- (3) For each $e > 0$, $t(Y, F_Y^{e*}\mathcal{G}, H) = p^e t(Y, \mathcal{G}, H)$.

Proof. This follows directly from the definition. \square

Proposition 4.6. Let Y be a normal projective variety of dimension n , let \mathcal{G} and H be as in Definition 4.4, and let Δ be an effective \mathbb{Q} -Weil divisor on Y such that $K_Y + \Delta$ is \mathbb{Q} -Cartier. Set $t := t(Y_0, \mathcal{G}|_{Y_0}, H|_{Y_0})$ for an open subset $Y_0 \subseteq Y$ satisfying $\text{codim}(Y \setminus Y_0) \geq 2$. If D is a Cartier divisor such that $D - (K_Y + \Delta) - nA + tH$ is ample for some base point free ample divisor A , then $\mathcal{G}^{**}(D)$ is generically globally generated.

Proof. Since $t(Y_0, \mathcal{G}|_{Y_0}, H|_{Y_0})$ is the supremum, there exists an $\varepsilon \in T(Y_0, \mathcal{G}|_{Y_0}, H|_{Y_0})$ such that $B := D - (K_Y + \Delta) - nA + \varepsilon H$ is ample. We fix such an ε . By the definition, $p^e \varepsilon \in \mathbb{Z}$ and there is a generically surjective morphism $\bigoplus \mathcal{O}_Y \rightarrow (F_Y^{e*}\mathcal{G})^{**}(-p^e \varepsilon H)$

for every $e \gg 0$. Let $l > 0$ be an integer such that $l(K_Y + \Delta)$ is Cartier and $l\varepsilon \in \mathbb{Z}$. For every $e \geq 0$, we denote by q_e and r_e respectively the quotient and the remainder of the division of $p^e - 1$ by l . Hence we have following generically surjective morphisms

$$\begin{aligned} & \bigoplus F_Y^e \mathcal{O}_Y(q_e l(B + nA) + (r_e + 1)(D - K_Y + \varepsilon H) + K_Y) \\ & \cong F_Y^e \bigoplus \mathcal{O}_Y(q_e l(B + nA) + (r_e + 1)(D - K_Y + \varepsilon H) + K_Y) \\ & \rightarrow F_Y^e ((F_Y^e)^* \mathcal{G})^{**} \otimes \mathcal{O}_Y(q_e l(B + nA - \varepsilon H) + (r_e + 1)(D - K_Y) + K_Y)) \\ & = F_Y^e ((F_Y^e)^* \mathcal{G})^{**} \otimes \mathcal{O}_Y(-q_e l\Delta + p^e(D - K_Y) + K_Y)) \\ & \cong (\mathcal{G}^{**}(D) \otimes F_Y^e \mathcal{O}_Y((1 - p^e)K_Y - q_e l\Delta))^{**} \\ & \rightarrow (\mathcal{G}^{**}(D) \otimes F_Y^e \mathcal{O}_Y((1 - p^e)K_Y))^{**} \\ & \rightarrow \mathcal{G}^{**}(D), \end{aligned}$$

where the isomorphism in the fifth line is induced by the projection formula, and the last morphism is induced by $\phi_Y^{(e)}$. Therefore, it is sufficient to show that

$$F_Y^e \mathcal{O}_Y(q_e l(B + nA) + (r_e + 1)(D - K_Y + \varepsilon H) + K_Y)$$

is globally generated for each $e \gg 0$. Since $0 \leq r_e < l$, by the Serre vanishing theorem, we have

$$\begin{aligned} & H^i(Y, \mathcal{O}_Y(-iA) \otimes F_Y^e \mathcal{O}_Y(q_e l(B + nA) + (r_e + 1)(D - K_Y + \varepsilon H) + K_Y)) \\ & \cong H^i(Y, F_Y^e \mathcal{O}_Y(q_e l(B + (n - i)A) + (r_e + 1)(D - K_Y + \varepsilon H - iA) + K_Y)) = 0 \end{aligned}$$

for each $i > 0$ and $e \gg 0$. Hence our claim follows from the Castelnuovo-Mumford regularity ([Laz04, Theorem 1.8.5]). \square

Proposition 4.7. *Let Y be a normal projective variety, let \mathcal{G} be a coherent sheaf, and let H be an ample divisor. If $t(Y_0, \mathcal{G}|_{Y_0}, H|_{Y_0}) \geq 0$ for some open subset $Y_0 \subseteq Y$ satisfying $\text{codim}(Y \setminus Y_0) \geq 2$, then \mathcal{G} is weakly positive.*

Proof. By the hypothesis and Lemma 4.5, we have

$$t(Y_0, (S^\alpha \mathcal{G})^{**}|_{Y_0}, H|_{Y_0}) \geq t(Y_0, \mathcal{G}^{\otimes \alpha}|_{Y_0}, H|_{Y_0}) \geq \alpha t(Y_0, \mathcal{G}|_{Y_0}, H|_{Y_0}) \geq 0$$

for every $\alpha > 0$. Applying the previous proposition, we obtain an ample Cartier divisor D such that $(S^\alpha \mathcal{G})^{**}(D)$ is generically globally generated for every $\alpha > 0$, which is our claim. \square

The following proposition will be used in Section 6.

Proposition 4.8. *Let Y be a smooth projective curve, let \mathcal{G} be a vector bundle on Y , and let H be an ample divisor on Y . If $t(Y, \mathcal{G}, H) \geq 0$, then \mathcal{G} is nef. Moreover, if $t(Y, \mathcal{G}, H) > 0$, then \mathcal{G} is ample.*

Proof. The first statement follows directly from Proposition 4.7 and Remark 4.3. We prove the second statement. By the hypothesis, $(F_Y^e)^* \mathcal{G}(-H)$ is generically globally generated for some $e > 0$, so it is a nef vector bundle since Y is a curve. This means that $F_Y^e \mathcal{G}$ is an ample vector bundle, hence so is \mathcal{G} . \square

5. MAIN THEOREM

In this section, we prove the main theorem (Theorem 5.1). As applications of the theorem, we prove weak positivity theorem for certain surjective morphisms of relative dimension zero, one and two (Corollaries 5.3, 5.4 and 5.5 respectively).

Theorem 5.1. *Let $f : X \rightarrow Y$ be a separable surjective morphism between normal projective varieties, let Δ be an effective \mathbb{Q} -Weil divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p , and let $\bar{\eta}$ be the geometric generic point of Y . Let H be an ample Cartier divisor on Y . Assume that*

- (i) $K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}$ is finitely generated in the sense of Definition 3.8, and
- (ii) there exists an integer $m_0 > 0$ such that

$$S^0(X_{\bar{\eta}}, \Delta_{\bar{\eta}}, am(K_{X_{\bar{\eta}}} + \Delta)_{\bar{\eta}}) = H^0(X_{\bar{\eta}}, am(K_{X_{\bar{\eta}}} + \Delta)_{\bar{\eta}})$$

for each $m \geq m_0$.

Then $f_*\mathcal{O}_X(am(K_X + \Delta)) \otimes \omega_Y^{-am}$ is a weakly positive sheaf for every $m \geq m_0$.

Proof. We first note that $X_{\bar{\eta}}$ is a $k(\bar{\eta})$ -scheme of pure dimension satisfying S_2 and G_1 , and that each connected component of $X_{\bar{\eta}}$ is integral by the Stein factorization and separability of $K(X)/K(Y)$. Let $d > 0$ be an integer satisfying $a|(p^d - 1)$.

Step1. In this step we reduce to the case where X and Y are smooth. Let H be an ample Cartier divisor on Y . By Proposition 4.7, it suffice to prove that $t(Y_0, (f_*\mathcal{O}_X(am(K_X + \Delta)))|_{Y_0} \otimes \omega_{Y_0}^{-am}, H|_{Y_0}) \geq 0$ for each $m \geq m_0$, where $Y_0 \subseteq Y$ is an open subset satisfying $\text{codim}(Y \setminus Y_0) \geq 2$. Hence, replacing X and Y by their smooth loci, we may assume that f is a dominant morphism between smooth varieties (the projectivity of f may be lost, but we will not use it).

We set $t(m) := t(Y, f_*\mathcal{O}_X(am(K_{X/Y} + \Delta)), H)$ for each $m > 0$.

Step2. We show that there exist integers $l, n_0 > m_0$ such that $t(l) + t(n) \leq t(l + n)$ for each $n \geq n_0$. By the hypothesis (i) and Lemma 3.9, $R(X_{\bar{\eta}}, a(K_{X_{\bar{\eta}}} + \Delta)_{\bar{\eta}})$ is a finitely generated $k(\bar{\eta})$ -algebra. Hence for every $l > m_0$ divisible enough there exists an $n_0 > m_0$ such that the natural morphism

$$H^0(X_{\bar{\eta}}, al(K_{X_{\bar{\eta}}} + \Delta)_{\bar{\eta}}) \otimes H^0(X_{\bar{\eta}}, an(K_{X_{\bar{\eta}}} + \Delta)_{\bar{\eta}}) \rightarrow H^0(X_{\bar{\eta}}, a(l + n)(K_{X_{\bar{\eta}}} + \Delta)_{\bar{\eta}})$$

is surjective. This shows that the natural morphism

$$f_*\mathcal{O}_X(al(K_{X/Y} + \Delta)) \otimes f_*\mathcal{O}_X(an(K_{X/Y} + \Delta)) \rightarrow f_*\mathcal{O}_X(a(l + n)(K_{X/Y} + \Delta))$$

is generically surjective, thus we have $t(l) + t(n) \leq t(l + n)$ by Lemma 4.5.

Step3. We show that $t(mp^{de} - a^{-1}(p^{de} - 1)) \leq p^{de}t(m)$ for each $e > 0$ and for each $m \geq m_0$. By the hypothesis (ii) and Observation 3.19 (III), there exist generically surjective morphisms

$$\begin{aligned} f^{(de)}_*\mathcal{O}_{X^{de}}(((am - 1)p^{de} + 1)(K_{X^{de}/Y^{de}} + \Delta)) &\xrightarrow{\alpha} f_{Y^{de}*}\mathcal{O}_{X_{Y^{de}}}(am(K_{X_{Y^{de}}/Y^{de}} + \Delta_{Y^{de}})) \\ &\cong F_Y^{de*}f_*\mathcal{O}_X(am(K_{X/Y} + \Delta)), \end{aligned}$$

where $\alpha := f_{Y^{de}*}(\phi_{(X, \Delta)/Y}^{(de)} \otimes \mathcal{O}_{X_{Y^{de}}}(am(K_{X_{Y^{de}}/Y^{de}} + \Delta_{Y^{de}})))$, and the isomorphism follows from the flatness of F_Y . Hence by Lemma 4.5, we have

$$t(mp^{de} - a^{-1}(p^{de} - 1)) \leq t(Y, F_Y^{de*}f_*(\omega_{X/Y}^{am}(am\Delta)), H) = p^{de}t(m).$$

Step4. We prove the theorem. Set $m \geq m_0$. If $am = 1$, then $t(1) \leq p^d t(1)$ by Step3, which gives $t(1) \geq 0$. Thus we may assume $am_0 \geq 2$. Let $q_{m,e}$ be the quotient of $mp^{de} - a^{-1}(p^{de} - 1) - n_0$ by l and let $r_{m,e}$ be the remainder for $e \gg 0$. We note that $q_{m,e} > 0$ since $m \geq m_0 \geq 2a^{-1} > a^{-1}$, and that $p^{de} - q_{l,e} \xrightarrow{e \rightarrow \infty} \infty$. Then

$$q_{m,e}t(l) + t(r_{m,e} + n_0) \stackrel{\text{Step2}}{\leq} t(mp^{de} - a^{-1}(p^{de} - 1)) \stackrel{\text{Step3}}{\leq} p^{de}t(m),$$

and so $c := \min\{t(r + n_0) \mid 0 \leq r < l\} \leq p^{de}t(m) - q_{m,e}t(l)$. By substituting l for m , we have $c \leq (p^{de} - q_{l,e})t(l)$ for each $e \gg 0$, which means $t(l) \geq 0$. Hence $c \leq p^{de}t(m)$ for each $e \gg 0$, and consequently $t(m) \geq 0$. This completes the proof. \square

Remark 5.2. There exists a fibration $g : S \rightarrow C$ from a smooth projective surface S to a smooth projective curve C such that $g_*\omega_{S/C}^m$ is not nef for any $m > 0$ [Ray78][Xie10, Theorem 3.6]. This fibration does not satisfy condition (ii) of Theorem 5.1. Indeed, the geometric generic fiber of g is a Gorenstein curve which has a cusp, hence by [GW77] it is not F -pure. Since the dualizing sheaf of a Gorenstein curve not isomorphic to \mathbb{P}^1 is trivial or ample, the claim follows from Examples 3.4 and 3.11.

Corollary 5.3. *Let $f : X \rightarrow Y$ be a surjective morphism between normal projective varieties, let Δ be an effective \mathbb{Q} -Weil divisor on X , and let $a > 0$ be an integer such that $a\Delta$ is integral. If f is separable and generically finite, then $f_*\mathcal{O}_X(a(K_X + \Delta)) \otimes \omega_Y^{-a}$ is weakly positive.*

Proof. Since f is generically finite, the natural morphism

$$f_*\mathcal{O}_X(aK_X) \otimes \omega_Y^{-a} \rightarrow f_*\mathcal{O}_X(a(K_X + \Delta)) \otimes \omega_Y^{-a}$$

is an isomorphism at the generic point of Y . Thus it is enough to show the case of $\Delta = 0$. Since the geometric generic fiber $X_{\bar{\eta}}$ is a reduced $k(\bar{\eta})$ -scheme of dimension zero, the assertion follows directly from Theorem 5.1. \square

Corollary 5.4. *Let $f : X \rightarrow Y$ be a separable surjective morphism of relative dimension one between normal projective varieties, let Δ be an effective \mathbb{Q} -Weil divisor on X , and let $a > 0$ be an integer not divisible by p such that $a\Delta$ is integral. Assume that $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is F -pure, where $\bar{\eta}$ is the geometric generic point of Y . If $K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}$ is ample (resp. $K_{X_{\bar{\eta}}}$ is ample and $a \geq 2$), then $f_*\mathcal{O}_X(am(K_X + \Delta)) \otimes \omega_Y^{-am}$ is weakly positive for each $m \geq 2$ (resp. $m \geq 1$). In particular, if every connected component of $X_{\bar{\eta}}$ is a smooth curve of genus at least two, then $f_*\omega_X^m \otimes \omega_Y^{-m}$ is weakly positive for each $m \geq 2$.*

Proof. Let $U \subseteq X$ be a Gorenstein open subset such that $\text{codim}(X \setminus U) \geq 2$ and that $a\Delta|_U$ is Cartier. Since $\dim(X \setminus U) \leq \dim X - 2 = \dim Y - 1$, $X \setminus U$ does not dominate Y . Thus there exists an open subset $Y_0 \subseteq Y$ such that $f|_{X_0} : X_0 \rightarrow Y_0$ is a Gorenstein morphism and that $a\Delta|_{X_0}$ is Cartier, where $X_0 := f^{-1}(Y_0)$. In particular, $X_{\bar{\eta}}$ is Gorenstein and $(a\Delta)_{\bar{\eta}}$ is Cartier. Thus the statement follows directly from Corollary 3.14 and Theorem 5.1. \square

Corollary 5.5. *Let $f : X \rightarrow Y$ be a separable surjective morphism of relative dimension two between normal projective varieties. Assume that every connected component of geometric generic fiber is a normal surface of general type with rational double point singularities, and $p \geq 7$. Then $f_*\omega_X^m \otimes \omega_Y^{-m}$ is weakly positive sheaf for each $m \gg 0$.*

Proof. We note that in this case $K_{X_{\bar{\eta}}}$ is finitely generated (cf. [Băd01, Corollary 9.10]). This follows from Corollary 3.23 and Theorem 5.1. \square

6. SEMI-STABLE FIBRATION

In this section, we discuss the weak positivity theorem in the case of a semi-stable fibration. We begin by recalling some definitions.

Definition 6.1. A projective k -scheme C of dimension one is said to be *minimally semi-stable* if:

- it is reduced and connected,
- all the singular points are ordinary double point, and
- each irreducible component which is isomorphic to \mathbb{P}^1 meets other components in at least two points.

Definition 6.2. Let X be a smooth projective surface, and let Y be a smooth projective curve. A fibration $f : X \rightarrow Y$ is called *semi-stable fibration* if all the fibers are minimally semi-stable.

Definition 6.3. A fibration $f : X \rightarrow Y$ is said to be *isotrivial* if for every two closed fibers are isomorphic.

Notation 6.4. Let $f : X \rightarrow Y$ be a semi-stable fibration whose geometric generic fiber $X_{\bar{\eta}}$ is a smooth curve of genus at least two.

In the situation of Notation 6.4, by [Kol90, 4.3 Theorem] or by Corollary 5.4, $f_*\omega_{X/Y}^m$ is a nef vector bundle for each $m \geq 2$. In the first part of this section, we give a necessary and sufficient condition in terms of f for these vector bundles to be ample (Theorem 6.8). In the second part, we consider the positivity of $f_*\omega_{X/Y}$.

Example 6.5. In the situation of 6.4, assume that f is isotrivial. Then there exists a finite morphism $\varphi : Y' \rightarrow Y$ from a smooth projective curve Y' such that there exists a commutative diagram

$$\begin{array}{ccccccc} Z & \xleftarrow{g} & Z \times_k Y' & \xrightarrow{\cong} & X_{Y'} & \longrightarrow & X \\ \downarrow & & \downarrow h & & \downarrow f_{Y'} & & \downarrow f \\ k & \longleftarrow & Y' & \xlongequal{\quad} & Y' & \xrightarrow{\varphi} & Y, \end{array}$$

where Z is a closed fiber of f [Szp79]. Let D be an effective divisor on X such that $(K_{X/Y} \cdot D) = 0$. Then since $\omega_{Z \times_k Y' / Y'} \cong g^*\omega_Z$, and since ω_Z is ample, each irreducible component of $D_{Y'}$ is a fiber of g , thus we have $D_{Y'} \sim g^*E$ for some effective divisor E on Z . Then for each integer $m \geq 1$, we have

$$\varphi^*f_*\omega_{X/Y}^m(D) \cong h_*\omega_{(Z \times_k Y')/Y'}^m(D_{Y'}) \cong h_*g^*\omega_Z^m(E) \cong H^0(Z, \omega_Z^m(E)) \otimes_k \mathcal{O}_{Y'},$$

and the right-hand side is trivial, so is $\varphi^*f_*\omega_{X/Y}^m(D)$. In particular, $f_*\omega_{X/Y}^m(D)$ is not ample for each $m \geq 1$.

In order to prove Theorem 6.8, we recall some results due to Szpiro and due to Tanaka.

Theorem 6.6 ([Szp79, Théorème 1]). *In the situation of Notation 6.4, assume that f is non-isotrivial. Then $\omega_{X/Y}$ is nef and big. Furthermore, an integral curve C in X satisfies $(\omega_{X/Y} \cdot C) = 0$ if and only if C is a smooth rational curve with $(C^2) = -2$ contained in a fiber.*

Theorem 6.7 ([Tan12, Theorem 2.6]). *Let X be a smooth projective surface, let B be a nef big \mathbb{R} -divisor whose fractional part is simple normal crossing, and let N be a nef \mathbb{R} -divisor which is not numerically trivial. Then there exists a real number $r(B, N) > 0$ such that*

$$H^i(X, K_X + \lceil B \rceil + rN + N') = 0$$

for each $i > 0$, for every real number $r \geq r(B, N)$, and for every nef \mathbb{R} -divisor N' such that $rN + N'$ is a \mathbb{Z} -divisor.

Theorem 6.8. *In the situation of Notation 6.4, let Δ be an effective $\mathbb{Z}_{(p)}$ -divisor on X . Assume that $\lfloor \Delta_{\overline{\eta}} \rfloor = 0$. Then the following conditions are equivalent:*

- (1) f is non-isotrivial or $(K_{X/Y} \cdot \Delta) > 0$.
- (2) $f_*\mathcal{O}_X(m(K_{X/Y} + \Delta))$ is ample for each $m \geq 2$ such that $m\Delta$ is integral.
- (3) $f_*\mathcal{O}_X(m(K_{X/Y} + \Delta))$ is ample for some $m \geq 2$ such that $m\Delta$ is integral.

In particular, $f_*\omega_{X/Y}^m$ is ample for each $m \geq 2$ if and only if f is non-isotrivial.

Proof. (2) \Rightarrow (3) is obvious. (3) \Rightarrow (1) follows from Example 6.5, thus we have only to prove (1) \Rightarrow (2).

Step1. We show that if f is isotrivial, then there exists an $\varepsilon_1 \in (0, 1) \cap \mathbb{Q}$ such that $K_{X/Y} + \varepsilon\Delta$ is ample for every $\varepsilon \in (0, \varepsilon_1) \cap \mathbb{Q}$. We may assume that $X \cong Z \times_k Y$ for a smooth projective curve Z of genus ≥ 2 , and $f : X \rightarrow Y$ is the second projection. By the assumption of (1), there is an irreducible component Δ_i of Δ such that $g(\Delta_i) = Z$, where $g : X \rightarrow Z$ be the first projection. Thus there exists an $\varepsilon_1 \in (0, 1) \cap \mathbb{Q}$, $(K_{X/Y} + \varepsilon\Delta \cdot C) > 0$ for every $\varepsilon \in (0, \varepsilon_1) \cap \mathbb{Q}$ and for every integral curve C in X . This means that $K_{X/Y} + \varepsilon\Delta$ is an ample \mathbb{Q} -divisor by the Nakai-Moishezon criterion.

Step2. Let H be an ample divisor on Y . We show that there exists an $\varepsilon_2 \in [0, \varepsilon_1) \cap \mathbb{Q}$ satisfying $H^1(X, m(K_{X/Y} + \varepsilon_2\Delta) - f^*H) = 0$ for each divisible enough $m > 0$. When f is isotrivial, this follows from Step 1 and the Serre vanishing theorem. We assume that f is non-isotrivial, and set $\varepsilon_2 := 0$. Since $K_{X/Y}$ is nef and big by Theorem 6.6, there exists an integer $n > 0$ such that $nK_{X/Y} - f^*(K_Y + H) \sim B$ for some effective divisor B . Let C be an integral curve in X satisfying $(B \cdot C) < 0$. Then we have

$$(f^*(K_Y + H) \cdot C) > (B + f^*(K_Y + H) \cdot C) = (nK_{X/Y} \cdot C) \geq 0,$$

so C dominates Y , hence $(K_{X/Y} \cdot C) > 0$ by Theorem 6.6. Replacing n by a larger one if necessary, we may assume that B is also nef and big. Then for every $m \gg 0$, we have

$$H^1(X, mK_{X/Y} - f^*H) = H^1(X, K_X + B + (m - n - 1)K_{X/Y}) = 0,$$

where the second equality follows from Theorem 6.7.

Step3. We show that $t(Y, f_*\mathcal{O}_X(m(K_{X/Y} + \varepsilon_2\Delta)), H) > 0$ for each divisible enough $m > 0$. There exists an exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_X(m(K_{X/Y} + \varepsilon_2\Delta) - f^*H - X_y) \rightarrow & \mathcal{O}_X(m(K_{X/Y} + \varepsilon_2\Delta) - f^*H) \\ \xrightarrow{\rho} & \mathcal{O}_{X_y}(m(K_{X/Y} + \varepsilon_2\Delta) - f^*H) \rightarrow 0 \end{aligned}$$

for every closed point $y \in Y$. By Step 2, since $H + y$ is ample, $H^0(X, \rho)$ is surjective for each divisible enough $m > 0$. This means that

$$f_*\mathcal{O}_X(m(K_{X/Y} + \varepsilon_2\Delta) - f^*H) \cong (f_*\mathcal{O}_X(m(K_{X/Y} + \varepsilon_2\Delta)))(-H)$$

is globally generated, and we have $t(Y, f_*\mathcal{O}_X(m(K_{X/Y} + \varepsilon_2\Delta)), H) \geq 1 > 0$.

Step4. Let a be an integer not divisible by p such that $a\Delta$ is integral. We show that $f_*\mathcal{O}_X(am(K_{X/Y} + \Delta))$ is ample for each $m \geq 2/a$. Set $\varepsilon_3 := 1/(p^b + 1)$ for some integer $b \gg 0$. Then, since $(X_{\bar{\eta}}, (1 + \varepsilon_3)\Delta_{\bar{\eta}})$ is F -pure by Fedder's criterion [Fed83], the proof of Theorem 5.1 shows that

$$t(Y, f_*\mathcal{O}_X(m(K_{X/Y} + (1 + \varepsilon_3)\Delta)), H) \geq 0$$

for each divisible enough $m > 0$. On the other hand, let $n_0 \geq 2$ be an integer such that

$$H^1(X_{\bar{\eta}}, cn((1 - \varepsilon_2 - \varepsilon_3)K_{X_{\bar{\eta}}} + (1 - \varepsilon_2 - \varepsilon_3 - 2\varepsilon_2\varepsilon_3)\Delta_{\bar{\eta}})) = 0$$

for some divisible enough integer $c > 0$ and each $n \geq n_0$. This means that

$$\mathcal{O}_{X_{\bar{\eta}}}(c(1 - \varepsilon_2)n(K_{X_{\bar{\eta}}} + (1 + \varepsilon_3)\Delta_{\bar{\eta}}))$$

is 0-regular with respect to $D_n := c\varepsilon_3n(K_{X_{\bar{\eta}}} + \varepsilon_2\Delta_{\bar{\eta}})$, where note that D_n is a very ample divisor on $X_{\bar{\eta}}$. Hence the morphism

$$\begin{aligned} H^0(X_{\bar{\eta}}, c(1 - \varepsilon_2)n(K_{X_{\bar{\eta}}} + (1 + \varepsilon_3)\Delta_{\bar{\eta}})) \otimes H^0(X_{\bar{\eta}}, D_n) \\ \rightarrow H^0(X_{\bar{\eta}}, c(1 - \varepsilon_2 + \varepsilon_3)n(K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}})) \end{aligned}$$

is surjective by the Castelnuovo-Mumford regularity [Laz04, Theorem 1.8.5]. From this, the morphism

$$\begin{aligned} f_*\mathcal{O}_X(c(1 - \varepsilon_2)n(K_{X/Y} + (1 + \varepsilon_3)\Delta)) \otimes f_*\mathcal{O}_X(c\varepsilon_3n(K_{X/Y} + \varepsilon_2\Delta)) \\ \rightarrow f_*\mathcal{O}_X(c(1 - \varepsilon_2 + \varepsilon_3)n(K_{X/Y} + \Delta)) \end{aligned}$$

is generically surjective. By Lemma 4.5 and Step3, we have

$$t(Y, f_*\mathcal{O}_X(c(1 - \varepsilon_2 + \varepsilon_3)n(K_{X/Y} + \Delta)), H) > 0.$$

From now on we use notation of the proof of Theorem 5.1. There exists an integer $l > 0$ divisible enough such that $t(l) := t(Y, f_*\mathcal{O}_X(l(K_{X/Y} + \Delta)), H) > 0$. By an argument similar to Step2 in the proof of Theorem 5.1, there exists a $h_0 > 0$ such that $t(l) + t(h) \leq t(l + h)$ for every $h \geq h_0$. By Corollary 3.14, we have

$$S^0(X_{\bar{\eta}}, \Delta_{\bar{\eta}}, am(K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}})) = H^0(X_{\bar{\eta}}, am(K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}))$$

for each $m \geq 2/a$. Thus by an argument similar to Step3 and 4 in the proof of Theorem 5.1, we have

$$q_{m,e}t(l) + t(r_{m,e} + h_0) \leq t(mp^e - a^{-1}(p^e - 1)) \leq p^e t(m),$$

for every $e > 0$ with $a|(p^e - 1)$. Here, $q_{m,e} > 0$ and $r_{m,e}$ respectively the quotient and the remainder of the devision of $mp^e - a^{-1}(p^e - 1) - h_0$ by l . Hence we obtain $t(Y, f_*\mathcal{O}_X(am(K_{X/Y} + \Delta)), H) > 0$ for each $m \geq 2/a$, which implies $f_*\mathcal{O}_X(am(K_{X/Y} + \Delta))$ is ample by Proposition 4.8. This completes the proof. \square

The following example shows that we can not drop the assumption $\lfloor \Delta_{\bar{\eta}} \rfloor = 0$ in Theorem 6.8.

Example 6.9. In the situation of Notation 6.4, let Δ be an effective \mathbb{Q} -divisor on X , and let C be a section of f . Taking blow-up, and replacing Δ and C by their proper transforms, we may assume that Δ and C are disjoint. Set $\Delta' := \Delta + C$. Then, for some integer $a > 0$ such that $a\Delta$ is integral and for each $m \geq 2/a$, there exists an exact sequence

$$0 \rightarrow \mathcal{O}_X(am(K_{X/Y} + \Delta') - C) \rightarrow \mathcal{O}_X(am(K_{X/Y} + \Delta')) \rightarrow \mathcal{O}_C \rightarrow 0.$$

This induces a nonzero morphism

$$f_*\mathcal{O}_X(am(K_{X/Y} + \Delta')) \rightarrow \mathcal{O}_Y,$$

since

$$H^1(X_{\bar{\eta}}, am(K_{X_{\bar{\eta}}} + \Delta'_{\bar{\eta}}) - C_{\bar{\eta}}) \cong H^0(X_{\bar{\eta}}, (1 - am)K_{X_{\bar{\eta}}} - am\Delta'_{\bar{\eta}} + C_{\bar{\eta}}) = 0.$$

Hence $f_*\mathcal{O}_X(am(K_{X/Y} + \Delta + C))$ is not ample for each $m \geq 2$.

Next, in the situation of Notation 6.4 we consider the positivity of $f_*\omega_{X/Y}$. This relate to the p -ranks of fibers. Here the p -rank of a smooth projective curve C is defined to be the integer $\gamma_C \geq 0$ such that p^{γ_C} is equal to the number of p -torsion points of the Jacobian variety of C . It is known that $\gamma_C = \dim_k S^0(C, \omega_C)$. Jang proved that if the geometric generic fiber $X_{\bar{\eta}}$ is ordinary, (i.e., the p -rank of $X_{\bar{\eta}}$ is equal to $\dim_k H^0(X_{\bar{\eta}}, \omega_{X_{\bar{\eta}}})$), then $f_*\omega_{X/Y}$ is a nef vector bundle [Jan08]. Raynaud implied that if every closed fiber is a smooth ordinary curve, then f is isotrivial [Szp81, THÉORÈME 5]. On the other hand, Moret-Bailly constructed an example of semi-stable fibration such that $f_*\omega_{X/Y}$ is not nef [MB81]. In this case, $X_{\bar{\eta}}$ is a smooth curve of genus two and of p -rank zero. As a generalization, Jang showed that if f is non-isotrivial and the p -rank of $X_{\bar{\eta}}$ is zero, then $f_*\omega_{X/Y}$ is not nef [Jan10]. We generalize these results to the case of intermediate p -rank (Theorem 6.13), based on the method in [Jan10].

Theorem 6.10 ([Jan10, Corollary 2.5]). *In the situation of Notation 6.4, let \mathcal{M} and \mathcal{T} be the free part and the torsion part of $R^1 f_{Y^1*} \mathcal{B}_{X/Y}^1$ respectively, where $\mathcal{B}_{X/Y}^1$ is the kernel of $\phi_{X/Y}^{(1)} \otimes \omega_{X_{Y^1}/Y^1}$ ($\phi_{X/Y}^{(1)}$ is defined in Section 2). Then there exists an exact sequence of \mathcal{O}_{Y^1} -modules*

$$0 \rightarrow \mathcal{M}^* \rightarrow f^{(1)}_* \omega_{X^1/Y^1} \xrightarrow{f_{Y^1*}(\phi_{X/Y}^{(1)} \otimes \omega_{X_{Y^1}/Y^1})} F_Y^* f_* \omega_{X/Y} \rightarrow \mathcal{M} \oplus \mathcal{T} \rightarrow 0.$$

Theorem 6.11 ([Szp79, Proposition 2]). *In the situation of Notation 6.4, f is isotrivial if and only if $\deg f_* \omega_{X/Y} = 0$.*

Theorem 6.12 ([LS77, 1.4. Satz]). *Let \mathcal{E} be a vector bundle on a smooth projective curve C . If $F_C^{e*} \mathcal{E} \cong \mathcal{E}$ for some $e > 0$, then there exists an étale morphism $\pi : C' \rightarrow C$ from a smooth projective curve C' such that $\pi^* \mathcal{E} \cong \bigoplus \mathcal{O}_{C'}$.*

Theorem 6.13. *In the situation of Notation 6.4, the function*

$$s(y) := \dim_{k(y)} S^0(X_y, \omega_{X_y})$$

on Y is lower semicontinuous. Furthermore, f is isotrivial if and only if $s(y)$ is constant on Y and $f_\omega_{X/Y}$ is nef.*

Proof. By Observation 3.19 (IV) we get the first statement (set $\Delta = 0$ and $R = K_{X/Y}$). We show that the second statement. If f is isotrivial, then obviously $s(y)$ is constant, and by Example 6.5, $f_*\omega_{X/Y}$ is nef. Conversely, we assume that $s(y)$ is constant on Y . This means that $\text{coker}(\theta^{(e)})$ is locally free for each $e \gg 0$, where

$$\theta^{(e)} := f_{Y^e*}(\phi_{X/Y}^{(e)} \otimes \omega_{X_{Y^e}/Y^e}) : f_*^{(e)}\omega_{X^e/Y^e} \rightarrow f_{Y^e*}\omega_{X_{Y^e}/Y^e} \cong F_Y^{e*}f_*\omega_{X/Y}.$$

Then there exists a commutative diagram

$$\begin{array}{ccccc}
& & \theta^{(e+1)} & & \\
& \nearrow & \downarrow & \searrow & \\
f_*\omega_{X/Y} & \xrightarrow{\theta^{(e)}} & F_Y^{e*}f_*\omega_{X/Y} & \xrightarrow{F_Y^{e*}\theta^{(1)}} & F_Y^{e+1*}f_*\omega_{X/Y} \\
& \searrow & \downarrow \text{im}(\theta^{(e)}) & \nearrow \alpha^{(e)} & \downarrow \beta^{(e)} \\
& & \text{im}(\theta^{(e)}) & & \text{im}(\theta^{(e+1)}) \\
& & \downarrow & & \downarrow \\
& & F_Y^*\text{im}(\theta^{(e)}) & & \text{im}(\theta^{(e+1)})
\end{array}$$

for each $e > 0$. When $e \gg 0$, since $\alpha^{(e)}$ is a surjective morphism between vector bundles of the same rank, it is an isomorphism. Furthermore, since $\beta^{(e)}$ is an inclusion between subbundles of $F_Y^{e+1*}f_*\omega_{X/Y}$ of the same rank, it is an isomorphism. Thus we have $F_Y^*\text{im}(\theta^{(e)}) \cong \text{im}(\theta^{(e)})$, in particular, by Theorem 6.12, $\deg(\text{im}(\theta^{(e)})) = 0$. Suppose that $f_*\omega_{X/Y}$ is nef. Then, Theorem 6.10 shows that $(\ker(\theta^{(1)}))^*$ is nef, hence that $(\ker(\alpha^{(e)}))^*$ is nef. Thus for every $e > 0$, the exact sequence

$$0 \rightarrow \ker(\alpha^{(e)}) \rightarrow \text{im}(\theta^{(e)}) \rightarrow \text{im}(\theta^{(e+1)}) \rightarrow 0$$

induces that

$$\deg(\text{im}(\theta^{(e)})) = \deg(\text{im}(\theta^{(e+1)})) + \deg(\ker(\alpha^{(e)})) \leq \deg(\text{im}(\theta^{(e+1)})).$$

From this we have $\deg(\text{im}(\theta^{(1)})) \leq \deg(\text{im}(\theta^{(2)})) \leq \dots \leq 0$, and hence

$$0 \leq \deg f_*\omega_{X/Y} = \deg(\ker(\theta^{(1)})) + \deg(\text{im}(\theta^{(1)})) \leq 0.$$

Consequently, Theorem 6.11 shows that f is isotrivial, which completes the proof. \square

7. IITAKA'S CONJECTURE

In this section, we consider Iitaka's conjecture under the following hypotheses:

Notation 7.1. Let $f : X \rightarrow Y$ be a fibration between smooth projective varieties, let Δ be an effective \mathbb{Q} -divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p , and let $\bar{\eta}$ be the geometric generic point of Y . Assume that

- (i) $K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}$ is finitely generated in the sense of Definition 3.8, and
- (ii) there exists an integer $m_0 > 0$ such that for every integer $m \geq m_0$

$$S^0(X_{\bar{\eta}}, \Delta_{\bar{\eta}}, m(aK_{X_{\bar{\eta}}} + (a\Delta)_{\bar{\eta}})) = H^0(X_{\bar{\eta}}, m(aK_{X_{\bar{\eta}}} + (a\Delta)_{\bar{\eta}})).$$

Here condition (i) and (ii) are the same as in Theorem 5.1. We first prove the case where Y is of general type based on the method in [Pat13].

Theorem 7.2. *In the situation of Notation 7.1, assume that Y is of general type. Then*

$$\kappa(X, K_X + \Delta) \geq \kappa(Y, K_Y) + \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}).$$

In the proof we use the argument similar to [Pat13, §4] and the proof of [Pat13, Theorem 1.7].

Proof. We may assume that $\kappa(X_{\bar{\eta}}, K_{\bar{\eta}} + \Delta_{\bar{\eta}}) \geq 0$.

Step1. Set $S' := \{\varepsilon \in \mathbb{Q} \mid \kappa(X, K_{X/Y} + \Delta - \varepsilon f^* H) \geq \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}) + \kappa(Y)\}$, where H is an ample divisor on Y . We show that S' is nonempty. By the assumption (i), there exists an integer $b > 0$ such that $R(X_{\bar{\eta}}, ab(K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}))$ is generated by $H^0(X_{\bar{\eta}}, ab(K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}))$. By projection formula, there exists an integer $c > 0$ such that $f_* \mathcal{O}_X(ab(K_{X/Y} + \Delta) + cf^* H)$ is globally generated. Thus for every $m > 0$, the natural morphism

$$\bigotimes^m f_* \mathcal{O}_X(ab(K_{X/Y} + \Delta) + cf^* H) \rightarrow f_* \mathcal{O}_X(abm(K_{X/Y} + \Delta) + cmf^* H)$$

is generically surjective and shows that $f_* \mathcal{O}_X(abm(K_{X/Y} + \Delta) + cmf^* H)$ is generically globally generated. This implies

$$\begin{aligned} & \dim_k H^0(X, abm(K_X + \Delta) + cmf^* H) \\ & \geq \dim_{k(\bar{\eta})} H^0(X_{\bar{\eta}}, abm(K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}})) + \dim_k H^0(Y, abmK_Y), \end{aligned}$$

Hence for $\varepsilon_0 := -c/(ab)$, we have $\kappa(X, K_X + \Delta - \varepsilon_0 f^* H) \geq \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}) + \kappa(Y)$.

Step2. Set $S := \{\varepsilon \in \mathbb{Q} \mid \kappa(X, K_{X/Y} + \Delta - \varepsilon f^* H) \geq 0\}$. We show that $\sup S = \sup S'$. Since $S \supseteq S'$ we have the inequality \geq . We show \leq . For an $\varepsilon \in S$, $K_{X/Y} + \Delta - \varepsilon f^* H$ is \mathbb{Q} -linearly equivalent to an effective \mathbb{Q} -divisor. Thus for every $0 < \delta \in \mathbb{Q}$ and $\varepsilon_0 \in S'$,

$$\begin{aligned} \kappa(X, (1 + \delta)(K_{X/Y} + \Delta) - (\varepsilon + \delta\varepsilon_0)f^* H) & \geq \kappa(X, \delta(K_{X/Y} + \Delta - \varepsilon_0 f^* H)) \\ & \geq \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}) + \kappa(Y). \end{aligned}$$

This implies $(\varepsilon + \delta\varepsilon_0)/(1 + \delta) \leq \sup S'$. Since $\lim_{\delta \rightarrow 0} (\varepsilon + \delta\varepsilon_0)/(1 + \delta) = \varepsilon$, we have $\varepsilon \leq \sup S'$, and hence $\sup S \leq \sup S'$.

Step3. We show that $\sup S \geq 0$. For simplicity of notation, we denote $f_* \mathcal{O}_X(am(K_{X/Y} + \Delta))$ by \mathcal{G}_m . By the proof of 5.1, we have $t(Y, \mathcal{G}_m, H) \geq 0$ for each $m \geq m_0$. We fix an $m \geq m_0$ such that $\mathcal{G}_m \neq 0$, where such m exists by the assumption that $\kappa(X_{\bar{\eta}}, \Delta_{\bar{\eta}}, K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}) \geq 0$. Let $d > 0$ be an integer such that $a|(p^d - 1)$. Then for every $\varepsilon \in T(Y, \mathcal{G}_m, H)$ there exists an $e > 0$ such that $p^{de}e \in \mathbb{Z}$

and $(F_Y^{de*}\mathcal{G}_m)(-p^{de}\varepsilon H)$ has a nonzero global section. On the other hand, since $X_{Y^{de}}$ is a variety, the natural morphism $F_{X/Y}^{(de)\#} : \mathcal{O}_{X_{Y^{de}}} \rightarrow F_{X/Y}^{(de)}_* \mathcal{O}_{X^{de}}$ is injective, which induces injective $\mathcal{O}_{Y^{de}}$ -module homomorphism

$$F_Y^{de*}\mathcal{G}_m \cong f_{Y^{de}*}\mathcal{O}_{X_{Y^{de}}}(am(K_{X_{Y^{de}}/Y^{de}} + \Delta_{Y^{de}})) \hookrightarrow f^{(de)}_*\mathcal{O}_{X^{de}}(amp^{de}(K_{X^{de}/Y^{de}} + \Delta)).$$

Note that the reducedness of $X_{Y^{de}}$ follows from the separability of f and the flatness of F_Y . From this

$$H^0(X, amp^{de}(K_{X/Y} + \Delta) - p^{de}\varepsilon f^*H) \neq 0,$$

and hence we have $(amp^{de})^{-1}p^{de}\varepsilon = (am)^{-1}\varepsilon \leq \sup S$, and so

$$0 \leq \frac{t(Y, \mathcal{G}_m, H)}{am} \leq \sup S.$$

Step4. We show the assertion. By the assumption and Step3, there exists an $\varepsilon \in S'$ such that $K_Y - \varepsilon H$ is linearly equivalent to an effective \mathbb{Q} -divisor. Then

$$\begin{aligned} \kappa(X, K_X + \Delta) &= \kappa(X, K_{X/Y} + \Delta + f^*K_Y) \\ &\geq \kappa(X, K_{X/Y} + \Delta + \varepsilon f^*H) \geq \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}) + \kappa(Y). \end{aligned}$$

This is our claim. \square

Next, we show that Iitaka's conjecture when Y is an elliptic curve (Theorem 7.6). To this end, we recall some facts about vector bundles on elliptic curves.

Theorem 7.3 ([Ati57, Oda71]). *Let C be an elliptic curve, and let $\mathcal{E}_C(r, d)$ be the set of isomorphism classes of indecomposable vector bundles of rank r and of degree d . Then the following conditions are satisfied:*

- (1) *For each $r > 0$, there exists a unique element $\mathcal{E}_{r,0}$ of $\mathcal{E}_C(r, 0)$ such that $H^0(C, \mathcal{E}_{r,0}) \neq 0$. Moreover, for every $\mathcal{E} \in \mathcal{E}_C(r, 0)$ there exists an $\mathcal{L} \in \text{Pic}^0(C) = \mathcal{E}_C(1, 0)$ such that $\mathcal{E} \cong \mathcal{E}_{r,0} \otimes \mathcal{L}$.*
- (2) *For every $\mathcal{E} \in \mathcal{E}_C(r, d)$,*

$$(\dim H^0(C, \mathcal{E}), \dim H^1(C, \mathcal{E})) = \begin{cases} (d, 0) & \text{when } d > 0 \\ (0, -d) & \text{when } d < 0 \\ (0, 0) & \text{when } d = 0 \text{ and } \mathcal{E} \neq \mathcal{E}_{r,0} \\ (1, 1) & \text{when } \mathcal{E} \cong \mathcal{E}_{r,0}. \end{cases}$$

- (3) *Let $\mathcal{E} \in \mathcal{E}_C(r, d)$. If $d > r$ (resp. $d > 2r$) then \mathcal{E} is globally generated (resp. ample).*
- (4) *([Oda71, Corollary 2.9]) When the Hasse invariant $\text{Hasse}(C)$ is nonzero, $F_C^*\mathcal{E}_{r,0} \cong \mathcal{E}_{r,0}$. When $\text{Hasse}(C) = 0$, $F_C^*\mathcal{E}_{r,0} \cong \bigoplus_{1 \leq i \leq \min\{r, p\}} \mathcal{E}_{\lfloor (r-i)/p \rfloor + 1, 0}$.*

Remark 7.4. Let C be an elliptic curve, and let $r_0 > 0$ be an integer. When $\text{Hasse}(C) = 0$, Theorem 7.3 (4) shows that there exists an $e > 0$ such that $F_C^{e*}\mathcal{E}_{r,0} \cong \bigoplus \mathcal{O}_C$ for each $r = 1, \dots, r_0$. When $\text{Hasse}(C) \neq 0$, Theorem 7.3 (4) and Theorem 6.12 show that there exists an étale morphism $\pi : C' \rightarrow C$ from an elliptic curve C' such that $\pi^*\mathcal{E}_{r,0} \cong \bigoplus \mathcal{O}_{C'}$ for each $r = 1, \dots, r_0$.

We also need the following theorem.

Theorem 7.5 ([Iit82, Theorem 10.5]). *Let $f : X \rightarrow Y$ be a surjective morphism between smooth complete varieties, let D be a divisor on Y , and let E be an effective divisor on X such that $\text{codim}(f(E)) \geq 2$. Then $\kappa(X, f^*D + E) = \kappa(Y, D)$.*

Theorem 7.6. *In the situation of Notation 7.1, assume that Y is an elliptic curve. Then*

$$\kappa(X, K_X + \Delta) \geq \kappa(Y) + \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}}).$$

Proof. **Step1.** Let $M \geq m_0$ be an integer. We show that there exists a finite morphism $\alpha : Y' \rightarrow Y$ from an elliptic curve Y' such that for each $m = m_0, m_0 + 1, \dots, M$

$$\alpha^* f_* \mathcal{O}_X(am(K_{X/Y} + \Delta)) \cong \mathcal{F}_m \oplus \left(\bigoplus_{\mathcal{L} \in S_m} \mathcal{L} \right),$$

where \mathcal{F}_m is an ample and globally generated vector bundle, and $S_m \subseteq \text{Pic}^0(Y')$. This claim follows from Theorem 5.1 and the lemma below.

Lemma 7.7. *Let \mathcal{E} be a nef vector bundle on an elliptic curve C . Then there exists a finite morphism $\pi : C' \rightarrow C$ from an elliptic curve C' such that $\pi^* \mathcal{E} \cong \mathcal{F} \oplus (\bigoplus_{\mathcal{L} \in S} \mathcal{L})$, where \mathcal{F} is an ample and globally generated vector bundle on C' , and $S \subseteq \text{Pic}^0(C')$.*

Proof of Lemma 7.7. By Theorem 7.3 (3), replacing \mathcal{E} by $F_Y^{e*} \mathcal{E}$, we may assume that $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{G}$ where \mathcal{F} is ample and globally generated, and \mathcal{G} is a direct sum of elements of $\bigcup_{r>0} \mathcal{E}_C(r, 0)$. Hence, Theorem 7.3 (1) and Remark 7.4 complete the proof. \square

Step2. We show that $\kappa(X_{Y'}, K_{X_{Y'}} + \Delta_{Y'}) = \kappa(X, K_X + \Delta)$. Obviously, we need only consider when α is separable and when α is purely inseparable. If $\alpha : Y' \rightarrow Y$ is separable then it is étale, thus so is $\alpha_X : X_{Y'} \rightarrow X$, in particular $X_{Y'}$ is a smooth variety. Hence the claim follows from Theorem 7.5, where we note that $K_{X_{Y'}} \sim K_{X_{Y'}/Y'} \sim (K_{X/Y})_{Y'} \sim (K_X)_{Y'}$. If $\alpha = F_{Y/k}^{(e)}$ for some $e > 0$, then there is a commutative diagram

$$\begin{array}{ccccc} X^e & & & & \\ \downarrow F_{X/Y}^{(e)} & \searrow F_{X/k}^{(e)} & & & \\ X_{Y^e} & \xrightarrow{(F_{Y/k}^{(e)})_X} & X_{k^e} & \xrightarrow{\cong} & X \\ \downarrow f_{Y^e} & & \downarrow f_{k^e} & & \downarrow f \\ Y^e & \xrightarrow{F_{Y/k}^{(e)}} & Y_{k^e} & \xrightarrow{\cong} & Y. \end{array}$$

Since X_{Y^e} is a variety (cf. [Pat13, Lemma 5.2]), we have injective morphisms $\mathcal{O}_{X_{k^e}} \rightarrow (F_{Y/k}^{(e)})_X^* \mathcal{O}_{X_{Y^e}} \rightarrow F_{X/k}^{(e)} \mathcal{O}_{X^e}$, which induce injective morphisms

$$H^0(X, am(K_X + \Delta)) \hookrightarrow H^0(X_{Y^e}, am(K_{X_{Y^e}} + \Delta_{Y^e})) \hookrightarrow H^0(X^e, amp^e(K_{X^e} + \Delta))$$

for every $m > 0$. Thus $\kappa(X, K_X + \Delta) = \kappa(X_{Y^e}, K_{X_{Y^e}} + \Delta_{Y^e})$ as claimed.

Step3. We complete the proof. Write $\mathcal{G}_m := f_* \mathcal{O}_X(am(K_X + \Delta))$. Let $l, n_0 > m_0$ be as in the proof of Theorem 5.1. By the above argument, we may assume that $\mathcal{G}_m \cong \mathcal{F}_m \oplus (\bigoplus_{\mathcal{L} \in S_m} \mathcal{L})$ for each $m \in \{l\} \cup \{n_0 + i\}_{1 \leq i < l}$, where \mathcal{F}_m is an ample and

globally generated vector bundle, and $S_m \subseteq \text{Pic}^0(Y)$.

Claim. The subgroup G of $\text{Pic}^0(Y)$ generated by S_l is a finite group.

Proof of the claim. Let $d, q_{l,e}, r_{l,e}$ be as in the proof of Theorem 5.1 for each $e \gg 0$. Set $S_l = \{\mathcal{L}_1, \dots, \mathcal{L}_h\}$. Then for each $i = 1, \dots, h$, there exist generically surjective morphisms

$$(\mathcal{F}_l \oplus \mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_h)^{\otimes q_{l,e}} \otimes (\mathcal{F}_{n_0+r_{l,e}} \oplus (\bigoplus_{\mathcal{L} \in S_{n_0+r_{l,e}}} \mathcal{L})) \cong \mathcal{G}_l^{\otimes q_{l,e}} \otimes \mathcal{G}_{n_0+r_{l,e}} \rightarrow \mathcal{G}_{l p^{de} + a^{-1}(1-p^{de})} \rightarrow F_Y^{de*} \mathcal{G}_l \rightarrow \mathcal{L}_i^{p^{de}}$$

as in the proof of Theorem 5.1. It follows that there exists a nonzero morphism $\mathcal{L}_1^{t_1} \otimes \dots \otimes \mathcal{L}_h^{t_h} \otimes \mathcal{L} \rightarrow \mathcal{L}_i^{p^{de}}$ for some integers $t_1, \dots, t_h \geq 0$ satisfying $\sum_{i=1}^h t_i = q_{l,e}$ and for some $\mathcal{L} \in \bigcup_{r=0}^{l-1} S_{n_0+r}$. Since this is a nonzero morphism between line bundles of degree zero on a smooth projective curve, this is an isomorphism, in particular $\mathcal{L} \in G$. For each $i = 1, \dots, h$ we denote \mathcal{L}_i^{-1} by \mathcal{L}_{i+h} for each $i = 1, \dots, h$, and for each $m > 0$ we set

$$G(m) := \{\bigotimes_{i=1}^{2h} \mathcal{L}_i^{m_i} \mid 0 \leq m_i \text{ and } \sum_{i=1}^{2h} m_i \leq m\} \subseteq G.$$

Let $c > 0$ be an integer satisfying $\{\mathcal{L} \in G \mid \mathcal{L} \text{ or } \mathcal{L}^{-1} \text{ is in } \bigcup_{r=0}^{l-1} S_{n_0+r}\} \subseteq G(c)$. Then by the above argument $\mathcal{L}_1^{p^{de}}, \dots, \mathcal{L}_{2h}^{p^{de}} \in G(q_{l,e} + c)$. Since $p^{de} > q_{l,e} + c$ for some $e \gg 0$, there exists an $N > 0$ such that $G = G(N)$, which is our claim. \square

By the claim, there exists an $n > 0$ such that $n_Y^* \mathcal{L} \cong \mathcal{L}^n \cong \mathcal{O}_Y$ for each $\mathcal{L} \in S_l$. Hence, replacing f by its base change with respect to n_Y , we may assume that \mathcal{G}_l is globally generated. Then, for each $b \gg 0$, \mathcal{G}_{bl} is generically globally generated, because the natural morphism $\mathcal{G}_l^{\otimes b} \rightarrow \mathcal{G}_{bl}$ is generically surjective as in the proof of Theorem 5.1. Thus we have

$$\begin{aligned} \dim_k H^0(X, abl(K_{X/Y} + \Delta)) &= \dim_k H^0(Y, \mathcal{G}_{bl}) \\ &\geq \dim_{k(\bar{\eta})} (\mathcal{G}_{bl})_{\bar{\eta}} = \dim_{k(\bar{\eta})} H^0(X_{\bar{\eta}}, bl(aK_{X_{\bar{\eta}}} + (a\Delta)_{\bar{\eta}})) \end{aligned}$$

for each $b \gg 0$, and so $\kappa(X, K_X + \Delta) \geq \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}} + \Delta_{\bar{\eta}})$. \square

There are some recent progress on Iitaka's conjecture in positive characteristic. Let $f : X \rightarrow Y$ be a fibration between smooth projective varieties, and let $X_{\bar{\eta}}$ be the geometric generic fiber. Chen and Zhang proved that

$$\kappa(X) \geq \kappa(Y) + \kappa(Z)$$

when f is of relative dimension one, where Z is the normalization of $X_{\bar{\eta}}$ [CZ13, Theorem 1.2]. They also proved that

$$\kappa(X) \geq \kappa(Y) + \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}})$$

when $\dim X = 2$ and $\dim Y = 1$ [CZ13, Theorem 1.3]. Patakfalvi showed that

$$\kappa(X) \geq \kappa(Y) + \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}})$$

when Y is of general type and $S^0(X_{\bar{\eta}}, \omega_{X_{\bar{\eta}}}) \neq 0$ [Pat13]. For a related result, see [Pat14, Corollary 4.6].

On the other hand, as a direct consequence of Theorem 7.2, Theorem 7.6, and Corollary 3.23, we obtain the following new result:

Corollary 7.8. *Let $f : X \rightarrow Y$ be a fibration from a smooth projective variety X of dimension three to a smooth projective curve Y . Assume that the geometric generic fiber $X_{\bar{\eta}}$ is a normal surface of general type with rational double point singularities, and $p \geq 7$. Then*

$$\kappa(X) \geq \kappa(Y) + \kappa(X_{\bar{\eta}}).$$

Proof. We note that in this case $K_{X_{\bar{\eta}}}$ is finitely generated (cf. [Băd01, Corollary 9.10]). Thus the result follows from Corollary 3.23 and Theorems 7.2 and 7.6. \square

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